Financial Crashes versus Liquidity Trap: the Dilemma of Nonconventional Monetary Policy*

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Abstract

This paper examines quantity-targeting monetary policy in a two-period economy with fiat money, endogenously incomplete markets of financial securities, durable goods and production. Short positions in financial assets and long-term loans are backed by collateral, the value of which depends on monetary policy. The decision to default is endogenous and depends on the relative value of the collateral to the loan. We show that Collateral Monetary Equilibria exist and prove a refinement of the Quantity Theory of Money that turns out to be compatible with the long-run non-neutrality of money. Moreover, only three scenarios are compatible with the equilibrium condition: 1) either the economy enters a liquidity trap in the first period ; 2) or a credible expansionary monetary policy accompanies the orderly functioning of markets at the cost of running an inflationary risk ; 3) else the money injected by the Central Bank fuels a financial inflation whose burst leads to a crash in the next period with nonzero probability. This dilemma of monetary policy enables us to assess the default channel affecting trades and production, hence to provide a rigorous foundation to Fisher's debt deflation theory.

Keywords: Central Bank, Liquidity trap, Collateral, Default, Deflation, Quantitative Easing, Debt-deflation.
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1 Introduction

The financial crisis of 2007 and its subsequent adverse effect on GDP deeply challenges the classical understanding of recessions provided by general equilibrium and RBC models. Indeed, while no recession preceded the large number of mortgage defaults, these mortgage defaults and their adverse effect on banks capital caused economic slowdown and the near financial meltdown. Subsequent pessimism in the banking sector resulted in credit crunch. The origin of the view of economic recessions as being *caused* by financial instability (rather than being their origin) can be traced back in 1933, when Irving Fisher advocated his debt-deflation theory of Great Depressions. He argued that over-indebtedness can precipitate deflation in future periods and subsequently liquidation of collateralized debt: Debt is denominated in constant nominal terms, whereas the value of the collateral that secures this debt depends on market forces and monetary aggregates in the economy.

Next, if financial instability can cause economic recession, this raises the question as to whether the monetary environment may have contributed to the excessive leveraging and risk-taking in financial markets. US and global monetary instances have been criticized as having been excessively expansionary in the last decade (Taylor, 2009). According to this view, monetary policy in the aftermath of the 2001 recession remained too lax for too long and this triggered asset-price inflation, primarily but not exclusively on the US housing market, and a generalized leverage boom. Had it followed more closely, say, the Taylor rule, so goes the argument, the Fed would have tightened rates faster, instead of lowering interest rates further to counter perceived deflation risks. Accordingly, short-term rates would have been higher between 2001 and 2005, making the subsequent burst less pronounced.

During the 2007–09 crisis, this latter line of reasoning, however, encountered the zero-bound problem: While the Taylor rule would have recommended a negative interest rate, this is not possible to achieve. Fresh thinking was prompted on the options still available when the interest rate cannot be lowered any further (see, e.g., Eggertson and Woodford (2003)). Drawing lessons from the Japanese experience from the 1990s, the Fed especially reached the conclusion that monetary policy can still be effective — which was the origin of the zero-interest-rate policy and other unconventional policies, such as qualitative and quantitative easing (Meier, 2009).

The purpose of this paper is to examine, within the simplest possible model, the systemic effects of such unconventional monetary policies on trades in commodities and assets, production and defaults within a finite horizon framework of fully flexible prices in a monetary economy with consumers and entrepreneurs.[1] Within such a framework, the following questions can be answered: Will an expansionary monetary policy necessarily lead to over-leverage and, eventually, financial instability? Must it lead to inflation on the commodity market? Under which condition can it fuel inflation on financial assets? If an expansionary policy does induce over-leverage, can the deleveraging process lead to a collapse of trades?

For this purpose, two frictions are introduced: First, traders face a cash-in-advance constraint for their trades both in consumption commodities and financial

[1] We restrict the production sector to entrepreneurs (who are the only owners of their firm) in order to avoid problems arising from the valuation of a firm in an incomplete stock market economy. For simplicity, the model is stated for two-periods but all our results go through for any (finite) horizon.

assets. That fiat money be the sole medium of exchange can easily be explained as an institutional answer to transaction costs[2]. Since transactions on financial markets are simultaneous to those on commodity markets, our cash constraint accounts for the financial motives for money demand underlined by Ragot (2012).[3]

Second, agents are allowed to (endogenously) default on their long-term loan obligations as well as on their financial promises.[4] Thus, the need for collateral to back loans and financial assets arises. In all other respects, we maintain the structural characteristics of general equilibrium analysis, i.e. optimizing behaviour, perfectly competitive markets and rational expectations.

For simplicity, we consider a two-period economy with finitely many states of Nature in the second period, finitely many types of households, a Central Bank and no private banking sector. The agents we shall consider engage into long-term borrowing to buy durable goods which they pledge as collateral to secure their loan. Simultaneously, they can trade collateralized financial assets in order to redistribute wealth across time and uncertainty. Loans and assets are non-recourse and there is no utility penalty for defaulting. Whenever the face value of a security's promise is higher than the value of the collateral, the seller of the security can choose to default, and her collateral is seized. Market incompleteness is central to our analysis, since agents cannot write comprehensive contracts in order to hedge the possibility of default. Finally, money is the sole medium of exchange and the quantity of inside money is set by the Central Bank.

Herein, we will show under what conditions the monetary policy can drive the economy to a state which is characterized by defaults on collateralized loan obligations and/or financial promises, and whose final GDP depends upon such defaults. We do not engage in a detailed discussion of optimal monetary policy, but rather propose default as an additional channel through which monetary policy can affect the real economy.

1.1 The monetary dilemma

We first prove existence of a collateral monetary equilibrium (Theorem 1) at a level of generality which, to the best of our knowledge, had not been reached in the previous literature. For this purpose, we slightly modify the definition of the measure, γ, of gains-to-trade first introduced by Dubey and Geanakoplos (2003a), so as to accommodate for default. Furthermore, we make use of a new hypothesis (HDD) on financial assets' deliveries, saying roughly that assets' returns grow at a polynomial speed with respect to macrovariables. All the financial derivatives we are aware of satisfy this restriction. It serves in proving that, even when markets become illiquid, assets' prices remain bounded. Finally, since default is allowed at equilibrium, we do

[2]See, e.g., Dubey and Geanakoplos (2003b).

[3]There, it is suggested, indeed, that a cash-in-advance constraint on the goods market is necessary to explain why many households hold only small amounts of money, while a cash-in-advance constraint on financial market explains why a few households hold large quantities of money. This last friction is thus required to explain why the distribution of money across households is much more similar, in the US, to the distribution of financial assets than to that of consumption levels—a puzzle for theories which solely link money demand to consumption.

[4]This contrasts with, e.g., Kehoe and Levine (1993), where the default penalties constrain borrowing in such a way that there is no equilibrium default.

not need outside money, $\sum_h m_h$, to be present in positive quantity.[5] Of course, the consequence is that, absent of outside money, the cost of money, r, solely depends upon the ratio between default and inside money. So that whenever no default occurs in the second period and absent of outside money, $r = 0$.

Money is non-neutral in our set-up, both in the short- and in the long-run. The quantity of inside money pumped in by the Central Bank influences the volume of trade and production both in the first period (short-term) and in the second period (long-term). This is illustrated by means of an example thoroughly studied in section 2. Behind existence and non-neutrality, we prove an analogue of the QTM (cf. 4.1), the consequence of which is that, if the Central Bank injects an unbounded amount of inside money (i.e., if $M \to +\infty$), the level of prices must increase, since the volume of trade is physically bounded.[6] Thus, despite the fact that money is non-neutral, our QTM enables us to partially recover a traditional wisdom: when "too much" money is already circulating, adding more money just fuels inflation in the long-run.

The main result of this paper, however, is a *full characterization* of equilibria. The argument driving this characterization (Theorem 2) can be informally stated as follows. The need to improve the efficiency of trades calls for an increase on the quantity of money injected in the economy by the Central Bank. Indeed, such an increase will typically reduce the cost of trading, r, hence provide more incentives to trade and produce. In our two-period set-up, however, the impact of such an increase of money also depends upon agents' expectations. If investors trust that there won't be "enough money" in the next period (relatively to the current one), then the economy enters a global liquidity trap: the short-term interest rate shrinks to zero (as the stock of money increases) while real cash balances hold by households increase with no effect on the real economy. This is, roughly speaking, the *liquidity trap scenario*. As economic agents share rational expectations, and would use money to purchase in period 0 if they had inflationary anticipations about the next period, we could equivalently rephrase this phenomenon by saying: if the Central Bank cannot commit to sufficiently increasing future stocks of Bank money so as to increase second period prices, then, anticipating deflation (or, at least, insufficient inflation on tomorrow's spot commodity prices), households will hold more and more money in their portfolio without any inducement from changing period 0 prices—which remain fixed even whenever the Central Bank keep injecting more money.[7]

The alternative goes as follows: If, on the contrary, households (rationally) expect the Central Bank to pump in enough money in the second period so as to increase prices in the second period, agents go on trading and producing. The impact of a current increase of monetary liquidity, however, can be twofold. If the leverage ratio

[5]Loosely speaking, default amounts to creating "outside money" endogenously. Notice, however, that, even when there is no outside money, our approach does not reduce to the approach favoured by Bloise et al. (2005). Indeed, there the seigniorage of the Central Bank is redistributed to the households, while here, it is not. By doing so, we remain closer to the modelling option adopted by Dubey and Geanakoplos (2003a,b) and their followers, although we can dispense with the introduction of outside money.

[6]Although there is no short sale constraint, trades of financial assets are bounded because of the scarcity of collateral.

[7]Existence of such a robust liquidity trap had been already shown by Dubey and Geanakoplos (2006b) in a model without default. Here, the liquidity trap is but *one* out of three possible scenarios that completely characterize the equilibrium set.

on financial markets is small[8], then a sufficiently large additional quantity of money will "grease the wheel of commerce" at the cost of running the risk to significantly raise prices. Indeed, as already noticed, any Quantity Theory equation implies that an expansionary policy must be driven with care as there exists a threshold above which it will but fuel inflation on the commodity market. This is the *inflationary scenario*. By contrast—and here comes our third scenario—if the Central Bank does not increase sufficiently the second-period stock of money and if the leverage ratio on the market for financial assets is high enough, most of the inside money injected in the first period will fuel inflation on assets and collaterals, eventually resulting in a financial crash in at least one state in the future (which occurs with positive probability) due to the deflationary impact of the (relatively) restrictive monetary policy. It turns out that, in the state where this third scenario occurs, agents completely default on their long-run loans.[9] This is the *crash scenario*.

The strength of our characterization is to prove that there is no escape road from these three stylized narratives: If there is no liquidity trap, and if second-period monetary policy is not "sufficiently" (in a sense to be made precise) expansionary with respect to the first-period injection of money, then there exists at least one second-period state where a crash occurs and no trade takes place. To put it differently, if a central banker wants to facilitate trades by injecting more money *and* to avoid both a liquidity trap and a financial crash, he needs to commit to the inflationary risk. Surprisingly enough, the argument underlying our Dilemma is quite simple: Suppose that the quantity, M_s, of inside money pumped in by the Central Bank in the second-period state, s, is bounded, and that effective trades occur in that state: any Quantity Theory equation then implies that spot prices in state, s, must be bounded. If this holds for every second-period state s, then, at equilibrium, both commodity and asset prices must be bounded as well: otherwise, some agent could sell a tiny part of a very expensive item in period 0, store the money and buy the whole economy in the second period. But since, the Central Bank, by assumption, pumps in an unboundedly growing stock of first-period money (be it on the short-run or the long-run monetary market), the boundedness of first-period prices will violate the Quantity Theory of Money (QTM) unless the short-term interest rate, r_0, hits the zero lower-bound. As a consequence, if no liquidity trap occurs in the first period, it must be that either (unbounded) inflation arises or trades vanish in some second-period state s. Since we confine ourselves to the (generic) class of economies for which there are positive gains to trade in every second-period state, the collapse of trades can only occur because of some brute deleveraging process, the symptom of which is that it must be accompanied with a complete default in the long-run monetary market. In a sense, what this paper does is simply to provide a micro-founded framework with rational expectations where this very simple story can be made precise.

In section 2, we exhibit a simple example where all the stylized facts scrutinized in this paper are at work. In particular, we show that the three scenarios already alluded to can occur. Regarding the scenario of a financial crash, the example suggests that the higher was inflation on financial markets, the deeper will be the crash in the bad state of the second period. What makes this phenomenon compatible with

[8]Equivalently, if margin requirements are high.

[9]This raises the question of whether such a crash could induce the bankruptcy of the Central Bank. This issue is left for further research.

our standard rational expectations framework is the assumption that investors share heterogeneous beliefs. The 2007–09 crisis highlighted the role of belief heterogeneity and how financial markets allow investors with different beliefs to leverage up and speculate. Several investment and commercial banks invested heavily in mortgage-backed securities, which subsequently suffered large declines in value. At the same time, some hedge funds took advantage from the securities by short-selling them. One reason for why there has been little attention paid to belief heterogeneity is the celebrated market selection hypothesis of Friedman (1953): On the long-run, there should be little differences in beliefs because agents with wrong beliefs should be driven out of the market by those who share correct beliefs.[10] As shown by Cao (2010), however, collateral requirements prevent the market forces from driving out investors with wrong beliefs. Since we consider collateralized assets and loans, we therefore believe it natural to consider investors with heterogeneous beliefs.

One by-product of our analysis is that, in the absence of a liquidity trap, what enables inflation over commodities to remain bounded despite the increase of liquidity, is the fact that most of the injected liquidity migrates towards financial markets, thus fuelling financial inflation. The example of section 2 below exhibits an economy where this happens for a high enough leverage ratio. This provides a theoretical narrative for the Great Moderation of inflation experienced by Western countries for the last two decades that contrasts sharply with the conventional wisdom. Furthermore, this example shows that a constant level of domestic prices is compatible with an increase of the quantity of money injected by the Bank, together with a huge inflation both on the financial markets and the market for collaterals. This happens, e.g., under the condition that the leverage ratio increases at the same speed as the quantity of money injected in period 0. The reason why collaterals are not immune against inflation is that they play a dual role: they are used both for the intrinsic value and as collaterals. The larger is financial inflation, the more attractive are collaterals.

1.2 Implications for monetary policy

Our results raise four kinds of questions:

1) Is the Central Bank condemned to an unbounded increase of money in our model? No: it can perfectly decide to inject few money in both periods. The only cost of such a "prudent" policy is that it results in a second-best allocation of resources and production plans, whose inefficiency could be partially removed by further creation of inside money. By injecting more money, however, the Central Bank runs the risk of entering into our Dilemma.

2) What happens if the Bank decides for a time-consistent expansionary policy? Theorem 2 says that two regimes can emerge at equilibrium: either the Bank's policy is not credible when it claims it will foster inflation tomorrow, and the economy enters a liquidity trap, or its policy is credible, markets function "normally" (enabling agents to exploit more gains-to-trade as the stock of period 0 money increases) but inflation will occur above a certain level of money injection. In our simplified two-period model, which regime (deflation *versus* inflation) will take place at equilibrium depends upon the way agents' expectations are anchored. The point, however, is that both expectations are rational and compatible with the equilibrium conditions.

[10] See also Blume and Easley (2006) and Sandroni (2000).

3) When does a crash occur? When, after having engaged into an expansionary path, the Central Bank does no lend as much money in the second period as it would have been expected to do, given its first-period behaviour. Indeed, most confident economic actors increased their indebtedness in the first period, so that if there is at least one second-period random state where the Bank's policy is not sufficiently expansionary, then the Fisherian debt-deflation effect takes place, and a crash occurs. Again, this is compatible with economic actors having rational confidence in the Bank's ability to inject sufficiently money as agents' expectations will be confirmed in most second period random states, but not all. The possibility of borrowing against the asset makes it possible for fewer investors to hold all the assets in the economy. Hence, the marginal buyer is someone more optimistic than when the leverage rate is lower, raising the price of the asset. This effect was first identified in Geanakoplos (1997). This connection between leverage and asset prices is precisely the Leverage Cycle theory discussed in Geanakoplos (2003) and Fostel and Geanakoplos (2008). Actually, a close look at the proof of our Dilemma (Theorem 2) shows that we don't *need* this leverage cycle theory for the Dilemma itself. We rather use it in order to construct an example (section 2) where a crash *effectively* happens.

4) Which ingredients are indispensable for the Dilemma to hold? Obviously, we need heterogeneous agents in order to be able to speak about debt: a representative agent model cannot exhibit an investor who owes a debt to nobody. Second, we need to introduce a demand for money in some way or another —and, here, this arises from our cash-in-advance constraint. In order to be able to conclude that an unbounded expansionary monetary policy leads to inflation, we also need the analogue of a QTM —and, again, the QTM, here, is nothing but the sum of individual liquidity constraints. For the crash scenario to occur, the Dilemma requires, moreover, that equilibrium be compatible with default. This, in turn, forces us to consider durable goods: If all goods were perishable, indeed, no contract would be traded since none of them could be backed with any commodity that is worth something in the second period. Finally, we need markets to be incomplete in some sense, otherwise even default would never occur as such at equilibrium: agents would be able to hedge themselves against the possibility of a credit event. Market incompleteness is also needed for the occurrence of the liquidity trap when no outside money enters the economy: Indeed, when markets are complete, absent of outside money, no liquidity trap can occur within our set-up (Corollary 5.2) .

Thus the model we present below seems to be the simplest form under which our Dilemma can be formulated. The unique ingredient that is superfluous for the Dilemma itself is the presence of financial assets subject to some endogenous incompleteness due to the collateral constraints. We could as well introduce exogenous market incompleteness or, even, close the financial sector. Thus, if she wishes so, when going through the paper, the reader can think of the financial asset market as being closed. Only in the illustrative example of section 2 do we use financial assets (collateralized by some durable capital) to exhibit a situation where the crash scenario occurs. To be more precise, Theorem 2 works under two environments: either there is some positive outside money or the long-run monetary market is open. Out of these two environments, the monetary Dilemma fails for obvious reasons: Indeed, absent of outside money and whenever the long-run market is closed, a collateral monetary equilibrium reduces to a barter collateral equilibrium, as defined, e.g., by Geanakoplos and Zame (2010). If, in addition, the financial asset market is closed, even the possibility of default disappears from the model.

5) One last ingredient at play in this paper is the assumption that investors' attitude towards uncertainty be heterogeneous—either because they share different beliefs about second-period uncertain states or because they exhibit diverse risk-aversions. This assumption is *not* needed for our Dilemma. However, we use it in the example of section 2, and it is handful to explain how the crash scenario can occur. Indeed, the role of heterogeneity in explaining how the burst of a financial inflation results into fire sales has long been acknowledged by Geanakoplos (2001): In "bullish" times, optimistic investors are those who are ready to purchase risky assets at the highest price, and for this purpose, who are more likely to borrow money. Pessimists (or more risk-adverse investors), on the contrary, rather belong to the lenders. When bad news occur, it is the shift of wealth between optimistic investors and pessimistic ones that can create a dramatic fall in prices and, eventually, a crash.

Before drawing the links between our approach and the literature , let us immediately mention which remedy can be thought of for this unfortunate dilemma which sheds some light about current debates on most Central Banks' non-conventional policies. We do not offer any magic alternate solution. However, our model points in the following direction. In order to avoid the threat of the liquidity trap, the Bank should convince economic actors that it will not tighten its monetary policy in the future. Long-term interest rates emerge as a good instrument for this purpose (see the discussion at the end of section 2). However, quantitative easing in the sense of outright asset purchasing can turn out to be ineffective in order to fight against the liquidity trap. A non-conventional policy designed to affect the yield curve at longer-than-usual horizons therefore seems unavoidable. On the other hand, in order to get rid of financial rational exuberance, regulatory authorities should reduce the leverage power of financial derivatives. What will be the upshot of such a policy mix? Since the reduction of leverage on financial markets will make them less attractive, the quantity theory of money implies that it will but induce domestic inflation on consumption commodities. Therefore, a purposely fostered inflation eventually emerges as a *pis aller* in order to avoid both deflation and financial crashes.

1.3 Related literature

Our work relates to the strand in the literature that argues that financial crises and in particular defaults on financial contracts can lead to economic recessions. Bernanke (1983) established that the Great Depression can be better explained when one explicitly models banking behaviour and introduces the concept of the balance sheet channel in the conduct of monetary policy. Bernanke and Gertler (1989) modelled, within a partial-equilibrium OLG framework, a collateral-driven credit constraint, introducing strong informational asymmetries, whereby the firm is only able to obtain fully collateralized loans. Hence, the value of the firm's assets has to be greater than the value of the loan or, at the limiting case, equal to. Due to the assumption of scarcity of assets and capital, the amount of credit to the firm shrinks in the presence of deflationary pressures on the prices of its assets. This introduces an external finance premium, which increases with a decrease in the relative price of capital. In turn, an increase in the cost of capital will result in a decrease in the marginal product and a reduction in GDP. They show that GDP and investment do not only depend on the fundamentals and productivity, but also on the soundness of the firms' financial situation, which is an important source of financial instability. We argue in this paper that, if informational asymmetries are certainly crucial element for the financial situation of economic agents to result in recessions, they are not the unique ones. Instead, the possibility of positive default and asset liquidation weaken the stability of the financial system and may result into unexploited gains-to-trade or a lower GDP. Indeed, as already stressed, belief (or informational) asymmetry plays in our Theorem 2 only for the occurrence of the third scenario. And, in fact, it could be replaced by asymmetries in risk-aversion. In any case, Theorem 2 shows that a liquidity trap may occur even absent of any kind of household or entrepreneurial heterogeneity.

But the emergence of our third, "crash" scenario shows that a Fisherian approach to debt can be recast within our perfectly competitive framework, as a decrease in the second-period money supply (relatively to first-period money supply) can lead to over-indebtedness, higher default and ultimately a crash in the financial sector. To our knowledge, the analytical framework supporting Fisher's theory had not yet been developed in the literature at the level of generality considered in this paper. The seminal work of Bernanke and Gertler (1989), which assumes fully collateralized loans, and Mendoza (2006), which introduces collateral constraints in an RBC model, have paved the road towards this direction. Nevertheless, they neither explicitly model money nor do they allow for positive default in equilibrium. Bernanke et al. (1999) consider a monetary economy along the lines of Dynamic New Keynesian models. However, they focus on real contracts and argue that the modelling of nominal ones is an important step for future research. This prevents their approach from shedding light into the observation that recessions follow financial crises. In the present work, nominal long-term loans play a crucial role, since their face value is invariant to deflationary pressures, while the value of collateral that backs them is not.

Relying on a standard New Keynesian framework, Eggertsson and Krugman (2010) also aim at providing firm theoretical grounds to the debt-deflation story. They prove indeed the existence of a liquidity trap induced by the deleveraging effect of an adversary shock on the debt-limit faced by borrowers. However, they deal with a cashless economy, where the Central Bank sets a nominal interest rate according to a Taylor rule, and where some firms face sticky prices. By contrast, we make the monetary transactions entirely explicit within a fully flexible price framework where the Central Bank can follow any arbitrary monetary policy. Above all, the authors introduce an exogenous debt limit faced by each consumer, set in real terms, and whose origin remains unmodelled. In a sense, the present paper provides an answer in terms of monetary policy: the exogenous shock to the debt limit introduced by Eggertsson and Krugman (2010) is replaced, here, by a contractionary monetary policy (or, more precisely, a policy that is rationally expected by investors not to be sufficiently expansionary).

Our approach is related to the work on the debt deflation theory of Sudden Stops (Mendoza, 2006, 2010; Mendoza and Smith, 2006). They introduce collateral constraints in an RBC model of a Small Open Economy to show that when debt is sufficiently high, an adverse productivity shock triggers the constraints and results in a fire-sales spiral, falling prices and a reduction in output. Our results point to the same direction, though contrary to them we consider a monetary economy with nominal contracts and focus on monetary shocks, which have not been studied in the literature as much. In addition, they do not allow for the possibility of default. The latter is crucial for our analysis, since it is the reason that capital gets reallocated to result in inefficient production. Due to fully flexible nominal prices, monetary policy only affects the price level in the final period and not the total output in the absence of default. However, default makes credit conditions more adverse and capital is not allocated efficiently. Thus, default may not be studied independently from credit and liquidity.

The closest work to ours is the path-breaking paper due to Lin et al. (2010). There, an entrepreneurial economy with two agents-firms is considered. One output commodity is produced by a single storable capital good. *Fiat* inside money is introduced but neither outside money, nor financial assets. The present paper can be viewed as the extension of Lin et al. (2010) to a multiple-commodity world, populated by an arbitrary finite number of households and/or entrepreneurs, some of them being possibly equipped with outside money, and where financial markets open in the first period. These additional features have several important consequences: As a result of the absence of outside money, Lin et al. (2010) cannot avoid the somewhat paradoxical property that, absent default, interest rates are all zero, even outside any liquidity trap. Moreover, they essentially consider the second-period impact of a contractionary monetary policy taking place in the first period. Consequently, they confine themselves to one (admittedly important) aspect of our scenario 2. Next, opening financial markets enables us to exhibit our third scenario. Moreover, Lin et al. (2010) do not exhibit the existence and robustness of a liquidity trap. By contrast, here, equilibrium interest rates are non-zero even in the absence of default, unless a liquidity trap has been reached.

The paper is organized as follows. The next section exhibits an example where the three scenarios of our Dilemma are present. Section 3 develops the general model. Section 4, we provide some general properties of Collateral Monetary Equilibria such as an endogenously determined quantity theory of money and the non-arbitrage relations within the yield curve. Section 5 is devoted to proving the alternative faced by monetary authorities: either they refrain from injecting money in the economy, at the cost of leaving an inefficient Collateral Monetary Equilibrium take place; or, they pump in a virtually infinite quantity of money but then, in order to escape from a liquidity trap, they must convince the economic actors that they will still inject a lot of money in the future. If they succeed in this non-conventional task, then they face two alternative risks: either a huge domestic inflation (when the leverage ratio on financial markets is low) or a financial rational exuberance (when the leverage ratio is high) whose burst may induce a global collapse of the economy in at least one second-period state of nature. A concluding section discusses the results of the paper in light of the 2007-09 crisis. Most of the proofs are placed in the Appendix. There, the double auction underlying our model is made explicit.

2 An example

The next example exhibits a situation where the three scenarios alluded to in the Introduction can occur. In particular, a global crash occurs with positive probability provided the leverage ratio is sufficiently large on financial markets.[11]

2.1 Some preparation

Let us briefly recall the basic properties of a one-shot Arrow-Debreu economy (with no uncertainty and no financial markets) where a cash-in-advance constraint, outside and inside money are introduced. The reader familiar with this literature can skip this subsection.

The cash-in-advance constraint looks like: $p \cdot z_h^+ \leq \mathbf{m}_h$, where p is the vector of prices of items available for trade, z_h is the vector of net trades of investor h, $z_h^{\ell +} := (\max\{z_h^\ell, 0\})$ its positive part (i.e., the net purchases of h) and $\mathbf{m}_h$ the cash balance available to h when entering the market. At equilibrium, this inequality will be binding, so that, summing over h yields the following version of the quantity theory of money (QTM):

$$p \cdot \sum_h z_h^+ = \sum_h \mathbf{m}_h. \tag{1}$$

The difference with Fisher's celebrated equation is that, here, money velocity is normalized to 1 while *both* prices, p, and the volume of transactions, $\sum_h z_h^+$, are endogenous, and depend upon the aggregate quantity of circulating money, $\sum_h \mathbf{m}_h$. The consequence of this endogeneity is that equation (1) is compatible with the non-neutrality of money. To be more precise, in a one-shot economy, a part, $\mu_h \leq \mathbf{m}_h$, of the cash balance is borrowed by h on the monetary market, and needs to be repaid, at the end of the day, at the cost of an interest rate, r. This cost induces a disincentive to trade unless the gains-to-trade are sufficiently large. In Dubey and Geanakoplos (2003a), it is proven that $\gamma > r$ is a sufficient condition for the existence of a monetary equilibrium with money having a positive value—where γ is a measure of the potential gains-to-trade at the initial allocation of endowments.[12] Another way to understand the interaction between the monetary and the real spheres of the economy is to realize that (1), together with the standard budget constraint, implies the following non-linear budget constraint:

$$p \cdot z_h^+ - \frac{1}{1+r} p \cdot z_h^- \leq m_h \tag{2}$$

where $m_h := \mathbf{m}_h - \mu_h$ is the outside money hold by h, and $z_h^- := z_h^+ - z_h$ is the vector of net sales of agent h.[13] Outside money can be interpreted in various alternative ways, such as: 1) fiscal injection ; 2) money free and clear of debt, arising from previous defaults that occurred in some non-modelled past[14]. Equation (2) highlights the role

[11]It can be seen as a monetary version of example 8 in Geanakoplos and Zame (2002).

[12]To be extended to our context in section 4.1 below.

[13]This formula will be extended below to our two-period set-up with collateralized assets and loans, see Proposition 4.3).

[14]We shall see in section 4 that, at the end of the last period of trades, all the outside money flows back to the Central Bank. Provided the Central Bank is linked with the government's Treasury, the interpretation of m^h as fiscal injection is compatible with the spirit of a non-Ricardian fiscal policy where public debt always vanishes at equilibrium, provided no default occurs on the long-run monetary market, but may not do so out of equilibrium or a soon as some default appears on the long loans of the Central Bank. Section 5 will prove that, once assets are collateralized, defaults occur, at equilibrium, in a number of circumstances. This yields some relevance to the second

of the interest rate, r, in the wedge between ask prices, p, and bid prices, $p/(1+r)$. The gains-to-trade condition, $\gamma > r$, can then be interpreted as meaning: potential gains-to-trade need to be sufficiently high with respect to the bid-ask spread for trades to be effective at equilibrium, and for money to have a positive value.

The link between monetary policy and equilibria rests on the distinction between outside money and inside money: $\mathbf{m}_h = m_h + \mu_h$. Since money is fiat, no agent has any interest in keeping money at the end of the day, so that all the outside money must be used to repay the interest of inside money, i.e.,

$$\sum_h m_h = r \sum_h \mu_h.$$

As a consequence, denoting by $M = \sum_h \mu_h$ the quantity of (inside) money injected by the Central Bank, one has, at equilibrium: $r = \sum_h m_h / M$. Thus, monetary policy —be it targeted towards the quantity of money, M, or towards the interest rate, r— will have an influence on the cost of trading. Since this cost must be balanced with respect to potential gains-to-trade for an equilibrium to emerge, monetary policy will, in general, influence not only the level of prices but also the volume of trades. Hence, money is non-neutral.

2.2 The monetary economy

There are two time periods ($t = 0, 1$), two states g (good) and b (bad) in the second period, two goods, F (food) and S (stock), and two agents P (the pessimist) and O (the optimist). Only the stock can serve as collateral. Food is perishable and stock is durable: food in period 0 must be consumed in that very period and cannot be inventoried, while stock can be consumed in period 0, stored into period 1 and consumed in this last period. Each agent has a specific storage (or production) function saying how much of a commodity she can store from period 0 to period 1. For simplicity, the storage functions are linear (constant returns to scale) $g_s^P(F, S) := (0, S)$ and $g_s^0(F, S) = (0, 2S)$, every F, S, s. In words, if the optimist stores one unit of stock, she gets two units of stock at the beginning of the second period: her productivity is twice that of the pessimist. Expected utility functions are:

$$u^P\big(F_0, S_0; (F_g, S_g), (F_b, S_b)\big) := F_0 + \tfrac{1}{2}(F_g + 10S_g) + \tfrac{1}{2}(F_b + 2S_b),$$
$$u^O\big(F_0, S_0; (F_g, S_g), (F_b, S_b)\big) := F_0 + .9(F_g + 10S_g) + .1(F_b + 6S_b).$$

Every agent is risk-neutral; optimists assess the good state as more likely and have higher marginal utility for the Stock in the bad state. Endowments are:

$$\mathbf{e}^P := \big((\mathbf{e}_{0F}^P, \mathbf{e}_{0S}^P); (\mathbf{e}_{gF}^P, \mathbf{e}_{gS}^P), (\mathbf{e}_{bF}^P, \mathbf{e}_{bS}^P)\big) = \big((40, 4); (40, 0), (40, 0)\big);$$
$$\mathbf{e}^O = \big((24, 0); (7, 0), (\tfrac{123}{14}, 0)\big).$$

The Pessimist is wealthy and owns 4 units of stock in period 0. No creation of stock takes place in period 1, so that all the stock available in that period must arise from some stock that has been stored in period 0. The quantity of outside money owned by h in state s is m_s^h. Monetary endowments are:

$$m^P := (m_0^P; m_g^P, m_b^P) = (2; 0, \tfrac{90}{21});$$
$$m^O = (2; 1, 1).$$

Agent O has the same monetary endowment in both second period states.

interpretation of outside money, in terms of money inherited from past defaults.

The Central Bank injects inside money M_s on the short-term loan market s. That is, the Bank precommits to the size of its borrowing or lending, letting interest rates be determined endogenously at equilibrium.[15] For the time being, monetary authorities do not intervene on the long-run market. A single asset can be traded in period 0, A_β, whose price is π^β, which promises delivery of β times the price of one unit of food in each state, collateralized by a unit of stock (hence, the collateral level of asset A_β is exogenous). The actual delivery of one unit of asset in state $s \in \mathbf{S}$ is

$$A_\beta^s := \min\{\beta p_{sF}; p_{sS}\}.$$

The parameter $\beta > 0$ is exogenous. The leverage ratio on A_β is then $\frac{\pi^\beta}{(1+r_0)p_{0S}-\pi^\beta}$.[16]

The timing of trades is as follows:

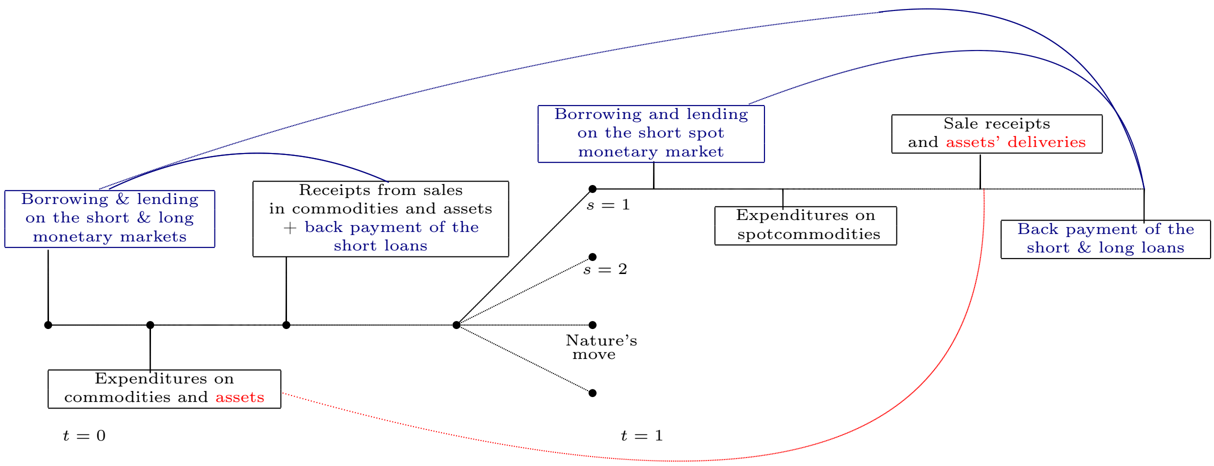

Fig 1. Timing of trades

2.3 Some preliminary remarks

1) Whatever being the quantity target of the Central Bank, $r_s \geq 0 \ \forall s \in \{0, b, g\}$, and $r_{\bar{0}} \geq r_0$. Indeed, if $r_s < 0$ in state s, the households could infinitely arbitrage the Bank. The same holds if $r_{\bar{0}} < r_0$ in the interest-rate target model: Investors could just finance any short-term loan by borrowing the corresponding cash on the long-run market. If, $r_{\bar{0}} < r_0$ in the quantity target model with $M_0 > 0$, then, nobody would borrow on the short-term loan market, which would therefore not clear.

[15] At the other extreme, we may suppose that the bank has interest rate targets, and precommits to supplying whatever money or bonds are demanded at those rates. In a one-shot economy both policies are equivalent (Dubey and Geanakoplos, 2003a), but already in a two-period economy with exogenous market incompleteness, they are not (Dubey and Geanakoplos, 2006a). Moreover, we are mainly interested in this paper in unconventional policies whose need arises from the zero-bound problem. We therefore confine ourselves to quantity target policies.

[16] Recall that an investor who borrows D and invests $A = K + D$ in an asset faces a leverage ratio $\ell := D/K$. Hence, our definition of the leverage. Equivalently, one could measure the "leverage" on loan markets through the margin requirement. In practice, margin requirements (which, in the US, are set by the Federal Reserve) are usually expressed in terms of a cash down payment as a fraction of the sale price. In this context, the margin requirement on A_β, is $\frac{(1+r_0)p_{0S}-\pi^\beta}{(1+r_0)p_{0S}}$.

2) When $r_s > 0$ $\forall s$, agents spend all the money at hand on purchases: Indeed, they can deposit money they do not intend to spend (or else borrow less), receiving the money back with interest, before they face the next buying opportunity.[17]

3) The optimist (i.e., the borrower or asset-seller) will always find it advantageous to buy stock entirely on margin because the cost of selling the asset in order to purchase the stock will never exceed the cost of the stock itself at time 1: Either the stock' price is higher than the return of the sold asset and the borrower earns the difference; or the stock's price is lower than the promised return, and the borrower's stock is seized and the borrower incurs no loss. Hence, at equilibrium, the number of units of security bought by agent P (i.e., the lender or, equivalently, the security-buyer) is equal to the number of units of stocks she sells.

4) In the good state, since agents have identical (linear) preferences, there is no trade as soon as $r_g > 0$. If, on the other hand, $r_g = 0$, then the equilibrium is indeterminate. For simplicity, we assume that $r_g = 0$. If either the Stock or the Food is actively traded in state g, both must be traded, so the equilibrium is interior and the relative price of the two goods will be $p_{gS}/p_{gF} = 10$. Indeed, suppose that agent h buys some shares and sells some food in state g. Then, denoting $\nabla^h_x := \frac{\partial u^h(\cdot)}{\partial x}$ the marginal utility of h for x, one must have:

$$\frac{\nabla^h_S}{p_{gS}} \geq \frac{\nabla^h_F}{p_{gF}}, \tag{3}$$

otherwise h would do better by reducing (by a little) both her purchase of Stock and her sale of Food. But since there are only two households, the reverse inequality must hold for h's partner. Next, if the equilibrium is interior (i.e., no consumption lies on a boundary), (3) holds as an equality.

5) For every agent h and every state s, let us denote by μ^h_s the marginal utility of income for h in state s:

$$\mu^h_s := \frac{\partial u^h(x^h)/\partial x_{s\ell}}{p_{s\ell}} \ \ell = 1, ..., L.$$

Note that μ^h_S is independent of which ℓ is used in its definition. Similarly, the marginal utility of h for income in date 0 is

$$\mu^h_0 := \frac{\partial u^h(x^h)/\partial x^h_{0\ell} + \sum_{s=1}^S \mu^h_s \left[p_s \cdot g_s(\delta^{0\ell})\right]}{p_{0\ell}}.$$

Again, μ^h_0 is independent from the commodity ℓ that served in defining it. Finally, the fundamental value of security j for agent h is defined by:[18]

$$\mathrm{FV}^h_j := \frac{\sum_{s=1}^S (p_s, 1) \cdot \mathbf{A}^j_s}{\mu^h_0}.$$

It follows from Geanakoplos and Zame (2009) that the price π^β of the asset A_β must weakly exceed its fundamental value. On the other hand, the pessimist is

[17]This is Lemma 2 in Dubey and Geanakoplos (2006a).
[18]As in Fostel and Geanakoplos (2008).

risk-neutral and indifferent about the timing of food consumption, so that the price, π^β, of the asset A_β cannot strictly exceed the pessimist's expectation of its delivery. Hence, the first-order condition yields:[19]

$$\frac{\nabla^P_{0F}}{p_{0F}} = \frac{1}{\pi^\beta}\sum_s \frac{1}{2}\mathbf{A}^\beta_s \frac{\nabla^P_{sF}}{p_{sF}}. \tag{4}$$

6) The relative value of collateral (stock) with respect to food in the good state being always 10 (cf. 4) above), the optimist will not default in the good state as long as $\beta \leq 10$. In the bad state, she defaults only if $\beta > p_{bS}/p_{bF} = 3$. Thus, whenever $3 < \beta \leq 10$, the relative price of A_β with respect to p_{0F} is $\frac{1}{2}(\beta + \frac{p_{bS}}{p_{bF}})$: in the good state, A_β delivers β units of Food ; in the bad state, 1 unit of stock (whose marginal utility is given by its price relative to Food):

$$\pi^\beta = \begin{cases} \beta p_{0F} & \text{if } \beta \leq 3 \\ \frac{p_{0F}}{2}(\beta + \frac{p_{bS}}{p_{bF}}) & \text{if } 3 < \beta \leq 10 \\ \frac{p_{0F}}{2}(\frac{p_{gS}}{p_{gF}} + \frac{p_{bS}}{p_{bF}}) & \text{if } \beta > 10. \end{cases}$$

2.4 The monetary dilemma

Case 1: low leverage

To fix ideas, suppose that $\beta = 1$. The unique collateral monetary equilibrium is then:

$p = \big(\pi^\beta, p_{0F}, p_{0S}; p_{gF}, p_{gS}; p_{bF}, p_{bS}\big) = \big(1, 1, 8; 1, 10; 1, 6\big)$, $M_0 = \frac{356}{7}$, and $M_b = \frac{195}{21}$,

$x^P = \big((64, \frac{4}{7}); (40, 0); (\frac{683}{14}, 0)\big)$,

$x^O = \big((0, \frac{24}{7}); (7, 0); (0, \frac{52}{7})\big)$.

$r_0 = 7/89 \sim 0.08$, $r_g = 0$ and $r_b = \frac{1}{2}$.

Let us explain why. The asset price is $\pi^\beta = 1$ and there is no default in period 1. At date 0, it is in the optimist's interest to sell as much food as possible in order to buy as much stock as possible. Hence, the optimist sells all her food and borrows to buy stock on the margin. But she cannot afford to buy all the stock; the pessimist keeps the remaining stock. The optimist's non-linear budget identity in state 0 is (see equation (2)):

$$\tilde{q}^O_{0S} - \frac{1}{1+r_0}(\pi^\beta \alpha^O_\beta + p_{0F} q^O_{0F}) = m^O_0$$

i.e.,

$$8\theta - \frac{1}{1+r_0}(\theta + 24) = 2,$$

that is, she buys θ units of stock and sells 24 units of food and θ units of asset. The pessimist's budget constraint is:

[19] Here, ∇^P_{sF} denotes the marginal utility of P with respect to Food in state s.

$$\tilde{\alpha}^P_\beta + \tilde{q}^P_{0F} - \frac{1}{1+r_0} p_{0F} q^P_{0F} = \theta + 24 - \frac{1}{1+r_0} 8\theta = 2,$$

Thus, the number of shares bought by the optimist in period 0 is $\theta = \frac{26+2r_0}{7+8r_0} = 24/7 < 4$. At the beginning of period 0, the optimist borrows $8\theta - 2 \sim 25.42$ on the short-loan market, and at the end, she spends her whole initial endowment in outside money in paying the interest of her loan, $r_0(8\theta - 2) = 2$. Therefore, the optimist does not save money from period 0 to period 1. The same holds for the pessimist. The aggregate quantity, $M_0 = 356/7$ is entirely borrowed by the two agents, and the payment of the interests, $r_0 M_0 = 4$, exhausts the aggregate outside money initially present at date 0.

In the bad state, the optimist spends $\theta p_{bF} = \theta$ to repay her risky loan on the asset market, gets an output of 2θ shares of stock out of the θ shares that had been held as collateral, spends $p_{bS}(4-\theta)$ to purchase the remaining $4-\theta$ shares of stock, and sells part of her initial endowment in food in order to finance her purchases. But her budget constraint is:

$$\theta + 6(4-\theta) - \frac{1}{1+r_b} q^O_{bF} = 1$$

which yields $q^O_{bF} = \frac{123}{14} = \mathbf{e}^O_{bF}$. Thus, having sold her entire endowment in food, the optimist has no additional income.

At the date 0 price vector of $(1, 8)$, the pessimist is exactly indifferent between stock and food. She fulfils her budget constraint in the bad state:

$$\mathbf{e}^O_{bF} - \frac{1}{1+r_b}\big(\theta + 6(4-\theta)\big) = \mathbf{e}^O_{bF} - \frac{2\theta}{1+r_b} = \frac{90}{21} = m^P_b$$

which confirms that the macrovariables listed above form indeed a CME.

Case 2: High leverage.

Now, take $3 < \beta \leq 10$.[20] We shall see that the unique equilibrium becomes:
$\pi^\beta = p_{0F} = p_{0S} = p_{gF} = p_{gS} = p_{bF} = 1, p_{bS} = 2, r_b = \frac{1}{2}$.
$x^P = \big((64, 0); (40, 0); (40, 4)\big)$
$x^O = \big((0, 4); (7, 0); (\frac{123}{14}, 4)\big)$.

Indeed, in period 0, the budget constraint of the optimist is

$$4p_{0S} - \frac{1}{1+r_0}(24 p_{0F} + 4\pi^\beta) = 2. \tag{5}$$

The budget constraint of the pessimist is

$$4\pi^\beta + 24 p_{0F} - \frac{4p_{0S}}{1+r_0} = 2, \tag{6}$$

The asset price is $\pi^\beta = \frac{p_{0F}}{2}(\beta + \frac{p_{bS}}{p_{bF}})$ (recall 11) *supra*) and $r_0 = \frac{4}{M_0}$.

Solving these equations yields:

$$p_{0S} = \frac{(1+r_0)}{2r_0}. \tag{7}$$

[20]Whenever $\beta > 10$, the (optimist) borrower will default in both states of period 1. Hence, A_β is identical to its collateral. The economy reduces to the situation where no asset can be traded.

$$p_{0F} = \frac{(1+r_0)}{r_0(14+\beta)}. \tag{8}$$

$$\frac{p_{0S}}{p_{0F}} = 7 + \frac{\beta}{2}. \tag{9}$$

$$\pi^\beta = \frac{(\beta+2)(1+r_0)}{2r_0(\beta+14)} \tag{10}$$

In the bad state, the optimist gets 8 units of stock out of the 4 hold as collateral, but she defaults on her financial promise, so that 4 units of stock are seized by the pessimist. Notice that, this time, the aggregate production of stock in the bad state is 8 while it was only $52/7 < 8$ in case 1. The monetary policy therefore has an impact not only on the final distribution of wealth but also on the GDP level. To put it more dramatically, the high leverage enables optimists (who, here, coincide with more productive agents) to borrow more money, hence to invest more and, finally, to produce more. This positive effect, however, is balanced by the fact that these more leveraged agents may encounter an adverse shock (here, the bad state) which forces a brute deleveraging process resulting into default and no-trade. The Japanese experience of the 1990s might be interpreted in the light of this very stylized example.

What happens, now, if the monetary policy becomes deliberately more expansionary? The leverage ratio is given by

$$\ell = \frac{q_\beta}{(1+r_0)p_{0S} - \pi^\beta} = \frac{\beta+2}{12+r_0(14+\beta)}.$$

For a fixed β, the impact of increasing the quantity of money, M_0, is transparent: as M_0 grows to infinity, r_0 shrinks to 0, $\ell \to (\beta+2)/12$, and both p_{0F}, p_{0S} and $\pi^\beta \to +\infty$. However, for this to be compatible with the equilibrium conditions, the quantity of money injected in period 1 must increase as well, at least in the good state. Indeed, suppose that $M_0 \to +\infty$ but M_g remains fixed. This means that prices in the good state will remain constant, say, equal to $p_g = (1, 10)$. But then, for M_0 high enough, the sale of a quantity, $\varepsilon > 0$, of food in period 0 will enable each agent to save enough money into period 1 to be able to buy the whole aggregate endowment of commodities in the good state. This contradicts the equilibrium condition. Thus, *either* M_g increases proportionately to M_0, so that prices in the good state also increase to infinity, *or* the economy falls into a liquidity trap in period 0. In the latter case, there is a threshold, $\overline{M}_0$, such that, for every $M_0 \geq \overline{M}_0$, the short-term interest rate hits its floor, $r_0 = 0$, and the additional money, $M_0 - \overline{M}_0$, is hoard by the agents at time 0, but remains unused (and flows back to the Central Bank at the end of period 0 at no cost).

Next, for a fixed M_0 (or, equivalently, a fixed r_0), if β increases, then the price of the asset, π^β increases together with ℓ, and p_{0S} remains constant while p_{0F} decreases. Thus, increasing the leverage ratio while keeping the quantity of circulating money constant induces a deflation on the domestic sector. This phenomenon could be called a "migration of liquidity towards the financial market", due to its increasing attractiveness.

Finally, if, say, $\beta = 1/r_0$, then π^β and p_{0S} still increase as β grows, but p_{0F} remains bounded. This suggests that, whenever the leverage ratio increases at a speed similar to that of the quantity of circulating money, then, this additional money fuels inflation on the financial market, but leaves domestic prices untouched (the price of food in period 0 is constant), while only the price of the collateral increases (as did the housing market prices between 2001 and 2006). This means that the deflationary effect due to the migration of liquidity towards financial markets can be compensated by a lax monetary policy. As for the leverage ratio,

$$\ell = \frac{\beta^2 + 2\beta}{13\beta + 14},$$

it is increasing in $\beta = M_0/4$. This might provide an explanatory scenario for the sequence of prices observed during 2001 and 2007.

2.5 Quantitative easing

In order to escape from the crux highlighted by the previous example (inflation/liquidity trap/crash), the Central Bank may engage in quantitative easing (as the Banks of England and Japan, and the Federal Reserve did after 2009). Recast in our set-up, such an unconventional policy consists in: either targeting the long-term interest rate, $r_{\overline{0}}$, or lending extra money by *buying* the asset A_β in period 0.

Let us begin with the first unconventional monetary policy.

Manipulating $r_{\overline{0}}$ clearly has an effect in our model, as soon as the long-term markets are active at equilibrium. This means that the usual explanation for the restriction of conventional policies to the short end of the yield curve—namely, that the determination of longer-term interest rates can be left to market mechanisms through no-arbitrage arguments—does not hold in our setting: equilibrium conditions do not enable, in general, to deduce $r_{\overline{0}}$ from $r_0, (r_s)_s$. As we shall see, more generally, in section 4, this is due to the collateral constraints, which break down the traditional non-arbitrage relationships within the yield curve. Thus, there is room for a policy that affects the yield curve at longer-than-usual horizons. No-arbitrage, however, does impose the following relationship within the yield curve in certain circumstances:[21]

$$r_{\overline{0}} \geq \min_{s \in \mathbf{S}} 1 + r_0 - \frac{1}{1 + r_s}$$

So that an increase of $M_{\overline{0}}$ must imply, in general, an increase of M_s in at least one state.[22] Thus, this first version of quantitative easing may succeed in circumventing the liquidity trap but at the cost of forcing the Central Bank to commit to an expansionary monetary policy in at least one future state.

Let us turn to the second interpretation of quantitative easing. To keep the analysis simple, suppose that the Central Bank does not offer money on the long-term market but rather offers to buy the asset A_β against fresh money.

Clearly, when $\beta > 2$, this would have no effect on the equilibrium: the optimist already borrows to the pessimist the needed amount of money in order to purchase the 4 units of stock available in period 0. Hence, the optimist holds already the maximal amount of collateral and there is no additional collateral to secure any additional loan.

[21] See Proposition (4.2) (iii) infra.

[22] In the previous example, this would mean in the bad state, as the good one is irrelevant.

When $\beta \leq 2$, the picture is more interesting. Absent such a quantitative easing policy, as we have already seen, the optimist cannot borrow enough from the sole pessimist to buy all the stock at time 0. Suppose, therefore, that the Central Bank buys A_β in place of the pessimist (who saves her money for a better use). If the quantity of fresh money thus injected is large enough, the optimist will now be able to buy *all* 4 shares of stock on margin at date 0. The budget identity of the optimist in period 0 is:

$$4p_{0S} - \frac{1}{1+r_0}(24p_{0F} + 4\beta p_{bF}) = m_0^2,$$

while the pessimist's budget constraint now is:

$$24p_{0F} - \frac{1}{1+r_0}4p_{0S} = m_0^1.$$

Solving these two equations yields:

$$p_{0F} = \frac{(1+r_0)^2}{24r_0(2+r_0)}\Big[\frac{4p_{bF}}{(1+r_0)^2} + \frac{m_0^2}{1+r_0} - m_0^1\Big]. \tag{11}$$

Thus, quantitative easing (in its second version) does have a real effect on the economy. Its weakness, of course, is that such a non-conventional policy is limited by the quantity of collateral already available in the economy. The Central Bank might therefore wish to supplement it with a more conventional policy consisting in reducing r_0 at the same time. Equation (11) shows that p_{0F} will then explode to infinity again. As already seen above, in order to circumvent the liquidity trap, such a mixed monetary policy needs to be accompanied by a commitment of the Central Bank to decrease r_s in each state of the world of period 1. In other words, at least within this example, the two understandings of quantitative easing given here do not suffice in circumventing the monetary dilemma.[23]

3 The general model

We consider a two-period monetary economy with heterogeneous agents, several consumption goods, money, capital markets and collateral constraints.

3.1 The physical economy

The set of states of nature is $\mathbf{S}^* := \{0, 1, ..., S\}$. State 0 occurs in period 0, then Nature moves and selects one of the states in $\mathbf{S} := \{1, ..., S\}$, which occurs in period 1. The set of commodities is $\mathbf{L} := \{1, ..., L\}$. Therefore, the commodity space[24] is $\mathbb{R}_+^{\mathbf{S}^* \times \mathbf{L}}$, where the pair $s\ell$ denotes commodity ℓ in state s.

[23] See McMahon and Polemarchakis (2011) for another work on quantitative easing in a GEI model, where, within a Ricardian framework (i.e., no outside money and full redistribution of the Bank's seigniorage), Quantitative Easing leads to the indeterminacy of prices.

[24] Throughout this paper, for a vector x of a real vector space $\mathbb{R}^n$, we denote $x > 0$ if x has non-negative components and at least one positive, and $x \gg 0$ if all its components are positive (we then write $x \in \mathbb{R}^n_{++}$). δ_ℓ is the vector $(0, ...0, 1, 0, ...0)$ of $\mathbb{R}^n_+$ where 1 stands in the ℓ^{th} coordinate.

The set of consumer types is $\mathbf{H} := \{1, ..., H\}$.[25] Each type h is endowed with $\mathbf{e}^h \in \mathbb{R}_+^{\mathbf{S}^* \times \mathbf{L}}$, and has a utility function: $u^h : \mathbb{R}_+^{\mathbf{S}^* \times \mathbf{L}} \to \mathbb{R}$. There is no loss of generality in assuming that, in each state, no agent has the null endowment and that every marketed good is actually present in the economy, i.e., $\mathbf{e}_s^h := (\mathbf{e}_{s1}^h, ..., \mathbf{e}_{sL}^h) > 0 \quad \forall h \in \mathbf{H}, s \in \mathbf{S}^*$, and $\sum_{h \in \mathbf{H}} \mathbf{e}_s^h \gg 0 \quad \forall s \in \mathbf{S}^*$. Each utility function, $u^h(\cdot)$, is continuous, quasi-concave, strictly increasing, and verifies the local non-satiation property: for each $x^h \in \mathbb{R}_+^{\mathbf{S}^* \times \mathbf{L}}$ and each $\varepsilon > 0$, there exists some y^h in the open ball, $B(x^h, \varepsilon)$, of radius ε and centred at x^h, such that $u^h(y^h) > u^h(x^h)$. No $u^h(\cdot)$ need to be separable.

If a bundle $x \in \mathbb{R}_+^{\mathbf{L}}$ is consumed (used) at time 0 by agent h, $g_s^h(x) \in \mathbb{R}_+^{\mathbf{L}}$ is what remains in state s at date 1. For each state s and every h, the storage mapping (it can be equivalently thought as a production function) $x \mapsto g_s^h(x)$ is assumed to be linear throughout this paper. Since utilities need not be monotone, certain durable assets can be interpreted as capital and the function $g_s^h(\cdot)$ as a random production technology faced by individual entrepreneurs.[26] Commodity ℓ is *perishable* for agent h at date 0 if $g_s^h(\delta_{0\ell}) = 0 \ \forall s$, and *durable* otherwise. Gold, residential mortgages, time purchases of cars and other consumer durables provide natural examples of such commodities.[27]

3.2 Money

Money is *fiat* but is the sole medium of exchange. Hence, all purchases are out of cash (this is the so-called cash-in-advance constraint (Clower, 1967). Money potentially enters the economy in two ways. Each agent h has endowments of money free and clear of debt, $m_s^h \geq 0$, in state $s \in \mathbf{S}^*$. We denote by $\overline{m}_s := \sum_h m_s^h$ the aggregate quantity of outside money available in state s. Observe that we do not impose $\overline{m}_s > 0$ in any state s. When positive, this monetary endowment can be interpreted as transfer payments from the Treasury government that are independent of equilibrium prices or, alternatively, as residual cash inherited from some default that occurred in the unmodelled past.[28] Following Woodford (2003) and Dubey and Geanakoplos (2003a,b), this is called *outside money*.

A Central Bank stands ready to make intra-period loans totalling $M_s > 0$ in each state $s \in \mathbf{S}^*$ and also to make the long loans totalling $M_{\overline{0}} > 0$ for two period starting at date 0. Money is perfectly durable. If the interest rate on loan $n \in N := \{\overline{0}, 0, 1, ..., S\}$ is r_n, then anyone can borrow $\mu_n/(1 + r_n)$ by promising to repay μ_n at the time the loan comes due.

In the initial period, agents finance their trading (or their investment in the capital good) both through short-term and collateralized long-term borrowing. For

[25]Each type of agent is thought of as represented by an interval, $[0, 1]$, of identical clones, with the Lebesgue measure. Hence, each agent takes macrovariables (prices and interest rates) as given. Throughout the paper, we focus on type-symmetric equilibria.

[26]Lin et al. (2010) provide a particular instance of this interpretation.

[27]*Storable* commodities that are not durable (such as tobacco, wine,...) are those goods that can be stored (i.e., $g_s^h(x) \neq 0$ in at least one state s) only if they are not consumed in period 0. In order to focus on the essential, we neglect such goods.

[28]Thus, in the parlance of Woodford (1994), when $\overline{m}_s > 0$ for some state s, we are considering non-Ricardian monetary policies. Since our purpose is not to study the optimal public policy, we take these transfers as exogenously given. As we shall see, equilibria exist whatever being the size of these transfers —even when they are zero.

simplicity, we assume that there is no default on the short-term loans market.[29] When agent h borrows from the collateralized long-term loan market,she pledges the goods purchased as collateral.[30] Only durable goods are eligible as collateral. In the first period, the borrower pays interest on her loans; she can never default on these, and this is consistent with no-default on the short-term loans market. In the second period, the borrower either delivers in full the amount of the collateralized loan or defaults. In case of default, the collateral pledged is foreclosed and is put for sale in the secondary market. The receipts are transferred to the Central Bank and determine the effective return on the collateralized loan. Between the two periods, the collateral is stored by the borrower.[31]

Let $\mathbf{C} \subset \mathbf{L}$ denote the non empty subset of durable commodities that are eligible as collaterals for long-term loans. We think of the collateral space $\mathbb{R}_+^{\mathbf{C}}$ as naturally embedded in the commodity space $\mathbb{R}_+^{\mathbf{L}}$ of period 0. At $t = 0$, agent h takes out a collateralized long-term loan $\mu_{\bar{0}}^h/(1 + r_{\bar{0}})$. The nominal interest rate is $r_{\bar{0}}$, and the interests are paid at the end of the period, whereas the principal is paid back in the next period.[32] Agent h pledges a basket of goods $\kappa_{\bar{0}}^h \in \mathbb{R}_+^{\mathbf{C}}$ as the collateral value of her loan $p_0 \cdot \kappa_{\bar{0}}^h = \mu_{\bar{0}}^h/(1 + r_{\bar{0}})$. She has bid enough commodities that make the collateral $\tilde{q}_{0\ell}^h \geq p_0^\ell \kappa_{\bar{0}\ell}^h \forall \ell \in \mathbf{C}$. At $t = 1$, the agent will deliver $\min\{p_s \cdot g_s^h(\kappa_{\bar{0}}^h) \ ; \ \mu_{\bar{0}}^h/(1 + r_{\bar{0}})\}$ in state s.

3.3 Collateralized financial assets

In addition to commodities and money, K financial assets can be traded in state 0, which deliver in the second period. The *macrovariables* are $\eta := (r, p, \pi)$, where:

$r \in \mathbb{R}_+^N :=$ interest rates on bank loans, $n \in N$.
$p \in \mathbb{R}_+^{\mathbf{S}^* \times \mathbf{L}} :=$ commodity prices, (p_s^ℓ).
$\pi := (\pi^1, ..., \pi^K) \in \mathbb{R}_+^{\mathbf{K}} :=$ the price of assets.

Sometimes, we write $\eta = (\eta(0), (\eta_s)_s)$, breaking η into its state components. Buying-and selling-nominal prices are identical. The *bid-ask spread* will be implicitly determined, at equilibrium, by the cost of borrowing *inside money* in order to facilitate purchase.[33]

All asset deliveries are supposed to be non-negative, and must be made in money. When the asset promise, $A_s^k = (a_{s1}^k, ..., a_{sL}^k, a_{sm}^k)$, includes commodities and money, the seller is asked to deliver the money equivalent to $(p_s, 1) \cdot A_s^k = \sum_\ell p_s^\ell a_{s\ell}^k + a_{sm}^k$,

[29] This is in conformity with current observation. On the Repo market, for instance, there is virtually no default, and even in crisis periods (such as 1994, 1998 or 2007–10), the rate of default remained hardly significant.

[30] Collateralized long-term loan extension is not an unusual function of modern central banks especially in the aftermath of the 2007 financial crisis. Alternatively, one could think of government sponsored institutions, which extend collateralized loans, e.g. Freddie Mac or Fannie Mae in the case of mortgages.

[31] As shown by Geanakoplos and Zame (2010), allowing for collateral to be warehoused or to be held and used by the lender creates only notational difficulties.

[32] If both interest and principal were to be paid back at the end of period 1, the demand for long-term loan in period 0 would not be bounded, even at equilibrium.

[33] Instead of imposing this cash-in-advance constraint—which is sometimes viewed as artificial, (Duffie, 1990)—we could as well start with a bid-ask spread. As long as the spread can be linked with the Central Bank's monetary policy (through equations akin to (42) and (43)), the two approaches (in terms of spread *versus* liquidity constraint) are equivalent in our setting.

where $p_s \in \mathbb{R}_+^{\mathbf{L}}$ is the spot commodity price in state s. Derivatives have pay-offs that depend upon the fundamental macrovariables (see *supra*). For example, a *call option* on firm j, with strike λ_j, pays off $(V_{sj} - \lambda_j)^+$ in each state $s \in \mathbf{S}$ (and usually, the strike is a function of some macrovariables). Another example is an inflation-indexed promise, which delivers $p_s \cdot \Lambda_s$ in state $s \in \mathbf{S}$, where $\Lambda_s \in \mathbb{R}_+^{\mathbf{L}}$ is a fixed basket of goods.

More generally, asset $k \in \mathbf{K} := \{1, ..., K\}$ promises pay-off $(p_s, 1) \cdot A_s^k(\eta_0, \eta_s)$ euros in each state $s \in \mathbf{S}$, where $A_s^k(\cdot, \cdot)$ is a continuous function of η_0 and η_s. We impose the following, fairly innocuous condition, which nevertheless seems to be new in the literature:

Hypothesis on Derivative Delivery (HDD)

For each k and s, $A_s^k(\cdot)$ is polynomially dominated at infinity by the macrovariables, i.e., $\exists b \geq 0$ with

$$A_s^k(\eta) = O(||(r,p)||^b),$$

where $||.||$ is a norm on the space of macrovariables.

Assumption HDD says that asset deliveries do not grow too fast in the sense that the quotient $A_s^k(\eta)/\,||(r,p)||^b$ remains bounded at infinity. It is verified if, say, the delivery function A_s^k is semi-algebraic—which is always the case in discrete-time finance industry.[34] In this formulation of HDD, we assume that asset deliveries are bounded in terms of other asset prices. Remark B) in the section 6 shows how to get rid of this restriction.

Agent h can buy or sell each asset k at price π^k. Because there are no a priori endowments of assets, their sales are "short sales". Notice that they are not a priori bounded.[35]

Agents can only sell the asset k if they hold shares of some collateral. Asset k is therefore associated with a vector, $\kappa^k \in \mathbb{R}_+^{\mathbf{L}}$, of collateral requirement.[36] If an agent sells one unit of asset k, she is required to hold κ_ℓ^k units of commodity ℓ as collateral.[37] Since the same commodity can be used as collateral for different financial assets, the agent is required to invest κ_ℓ^k in ℓ for each $k \in \mathbf{K}$. This means that tranching is not allowed. For simplicity also, only commodities are eligible as collaterals. In particular, we do not allow assets to be used as collaterals (no pyramiding).

[34]If only for computational purposes, the exponential function (which fails to verify (HDD)) is always replaced, at some point or another, by a polynomial approximation.

[35]In general equilibrium theory with exogenous incompleteness (see, e.g., Geanakoplos (1990)), this unboundedness destroys the existence of financial equilibria. The addition of money, however, suffices to restore existence Dubey and Geanakoplos (2003a).

[36]By contrast with the institutional arrangement for collateralized long-term loans, here, the vector κ^k is exogenous. This is the formulation used in most of the literature devoted to default in cashless economies, while the formulation we adopted for long-term loans (where the collateral vector is endogenous) has been introduced by Lin et al. (2010). As mentioned we could adopt the endogenous formulation for financial assets as well. We chose the exogenous one in order to show the flexibility of our approach.

[37]For the sake of simplicity, we do not allow the collateral to be held by the lender or to be warehoused (see Geanakoplos and Zame (2002)).

The return of asset k is the minimum between the total value of collateral and the promise at that state:

$$\mathbf{A}_s^k(\eta_0, \eta_s) := \min\Big\{(p_s, 1) \cdot A_s^k(\eta_0, \eta_1)\ ,\ p_s \cdot g_s^h(\kappa^k)\Big\} \tag{12}$$

Because of the scarcity of collaterals, collateral requirements introduce an endogenous bound on short sales. When $\kappa_\ell^k = 0$ for each k, ℓ, there is no collateral requirement, hence short sales are not limited. Notice that, at variance with Kiyotaki and Moore (1997), but in accordance with Geanakoplos and Zame (2010), we make the natural assumption that date 0 consumptions include goods not pledged as collaterals.

3.4 Liquidity constraints

The sequence of events is as follows. In period 0, agents borrow money either from the stock of outside money put on the loan markets by agents or from the Bank. There are two loan markets: one for the short term—where the Bank injects the stock, M_0, of inside money—and one for the long-term—where the Bank injects $M_{\overline{0}}$. On each market, an interest rate emerges (resp. r_0 and $r_{\overline{0}}$) so as to clear the money market. Next, the capital markets meet for the trade of financial assets, followed by the commodity markets. After this, there is a move of chance and the economy enters one of the state $s \in \mathbf{S}$ in period 1. In any state $s \in \mathbf{S}$, there is a fresh disposal of outside money and of Bank money M_s at an interest rate r_s. Money markets in state s are followed by another round of trade in spot commodities. Then, all the deliveries take place simultaneously: agents deliver on the asset they sold. Finally agents settle their debts with the Bank and the Bank with the private lenders.[38]

For any fixed choice of macrovariables η, let us now describe the set Σ_η^h of feasible choices of $h \in \mathbf{H}$, and the outcome that accrues to h as a function of η and of her strategy, $\sigma^h \in \Sigma_\eta^h$.

We denote:
$\mu_n^h :=$ IOUs (or Bank bonds) sold by h (h borrows $\mu_n^h/(1+r_n)$ on the loan market n)
$\kappa_{\overline{0}}^h :=$ collateral of long-term loans pledged by h
$\alpha_k^h :=$ asset $k \in \mathbf{K}$ sold by h
$q_{s\ell}^h :=$ commodity ℓ sold by h in state $s \in \mathbf{S}^*$.

A tilde on any variable will denote the money spent on it, i.e.,

$\tilde{\mu}_n^h :=$ money deposited (money spent on Bank bonds of type n) by h
$\tilde{\alpha}_k^h :=$ money spent by h in asset $k \in \mathbf{K}$
$\tilde{q}_{s\ell}^h :=$ bid of h on ℓ in state $s \in \mathbf{S}^*$.

The choice

$$\sigma^h := \Big((\mu_n^h, \tilde{\mu}_n^h)_{n \in N}, (\alpha_k^h, \tilde{\alpha}_k^h)_{k \in \mathbf{K}}, (q_{s\ell}^h, \tilde{q}_{s\ell}^h)_{s \in \mathbf{S}^*, \ell \in \mathbf{L}}, (\kappa_{\overline{0}\ell}^h)_{\ell \in \mathbf{C}}\Big) \geq 0$$

must satisfy a set of liquidity and physical constraints.

Period 0. The choice σ^h must satisfy the following *liquidity constraints*:

[38] To keep the anonymity of markets, all transactions on short-term or long-term loans pass through the Bank.

(i) Bank deposits in period 0 $\leq$ money endowed with:

$$\tilde{\mu}_0^h + \tilde{\mu}_{\overline{0}}^h \leq m_0^h. \tag{13}$$

(ii) Expenditures on commodities and assets $\leq$ money left in (13) + money borrowed on short- and long-term loans:

$$\sum_\ell \tilde{q}_{0\ell}^h + \sum_k \tilde{\alpha}_k^h \leq \Delta(13) + \frac{\mu_0^h}{1+r_0} + \frac{\mu_{\overline{0}}^h}{1+r_{\overline{0}}} \tag{14}$$

With expenditures from money borrowed on long-term loans comes the collateral requirement:

$$p_0 \cdot \kappa_{\overline{0}}^h = \frac{\mu_{\overline{0}}^h}{1+r_{\overline{0}}}, \quad \tilde{q}_{0\ell}^h \geq p_0^\ell \kappa_{\overline{0}\ell}^h \qquad \forall \ell \in \mathbf{C} \tag{15}$$

Finally, σ^h must verify the following *budget constraints:*

(iii) Money repaid (or received)[39] on loan 0 $\leq$ money left in (14) + money received from commodity sales + money obtained from sales of financial assets:

$$\mu_0^h - (1+r_0)\tilde{\mu}_0^h \leq \Delta(14) + p_0 \cdot q_0^h + \pi \cdot \alpha^h \tag{16}$$

(iv) Interests on loan $\overline{0}$ (repaid or received) $\leq$ money left in (16)

$$r_{\overline{0}} \frac{\mu_{\overline{0}}^h}{1+r_{\overline{0}}} - r_{\overline{0}} \tilde{\mu}_{\overline{0}}^h \leq \Delta(16) \tag{17}$$

Period 1. Similarly, in each state $s \in \mathbf{S}$ of period 1, we must have:

$(\text{v})_s$ Bank deposits in state s $\leq$ money inventoried from period 0 + fresh endowment of outside money (if any[40]):

$$\tilde{\mu}_s^h \leq \Delta(17) + m_s^h \tag{18}$$

$(\text{vi})_s$ Expenditures on commodities $\leq$ money left in (18) + money borrowed on loan s:

$$\sum_\ell \tilde{q}_{s\ell}^h \leq \Delta(18) + \frac{\mu_s^h}{1+r_s}. \tag{19}$$

$(\text{vii})_s$ cash needed for delivering on assets$\leq$ money left in (19) + money obtained from commodity sales:

$$\sum_k \mathbf{A}_s^k(\eta_0, \eta_s)\alpha_k^h \leq \Delta(19) + p_s \cdot q_s^h. \tag{20}$$

$(\text{vii})_s$ Money repaid on short-term loan s and long-term $\overline{0}$ $\leq$ money left in (20) + money obtained from asset deliveries:

[39] That is, there is netting on loan.

[40] Observe that we do not impose that initial endowments of outside money in the second period be positive.

$$\left[\min\left\{p_s \cdot g_s^h(\kappa_{\overline{0}}^h) \; ; \; \frac{\mu_{\overline{0}}^h}{1+r_{\overline{0}}}\right\} - \tilde{\mu}_{\overline{0}}^h\right] + \mu_s^h - (1+r_s)\tilde{\mu}_s^h \leq \Delta(20) + \sum_k \mathbf{A}_s^k(\eta_0, \eta_s)\frac{\tilde{\alpha}_k^h}{\pi^k}. \quad (21)$$

The choice σ^h must as well follow a set of *physical constraints*.[41] In period 0:

$$q_{0\ell}^h \leq \mathbf{e}_{0\ell}^h \quad \forall \ell \in \mathbf{L}. \qquad (22)$$

This condition means that the total amount of commodities sent to the clearing house cannot exceed the quantity of commodities at hand.

The consumption that accrues to h in period 0 is $x_0^h \in \mathbb{R}_+^{\mathbf{L}}$:

$$x_{0\ell}^h := \mathbf{e}_{0\ell}^h - q_{0\ell}^h + \frac{\tilde{q}_{0\ell}^h}{p_0^\ell} - \kappa_{\overline{0}\ell}^h - \sum_k \kappa_\ell^k \alpha_k^h \quad \forall \ell \in \mathbf{L}, \qquad (23)$$

The condition $x_{0\ell}^h \geq 0$ means that the initial endowment plus the net trade of h exceed the quantity hold as collateral (for long-term loans and financial assets[42]).

The physical constraints in period $s \in \mathbf{S}$ are:

$$q_{s\ell}^h \leq \mathbf{e}_{s\ell}^h + g_{s\ell}\big(\kappa_{\overline{0}}^h + \sum_k \kappa^k \alpha_k^h\big) + g_{s\ell}(x_0^h) \quad \forall s, \ell \in \mathbf{S} \times \mathbf{L}, \qquad (24)$$

This condition says that the total amount of commodities supplied in state s in the second period cannot exceed the initial endowment + the collateral stored from period 0 + what remains from the bundle consumed at time 0.

The final consumption in s is:

$$x_{s\ell}^h := \mathbf{e}_{s\ell}^h + g_s^h\big(\kappa_{\overline{0}}^h + \sum_k \kappa_\ell^k \alpha_k^h\big) + g_s^h(x_0^h) - q_{s\ell}^h + \frac{\tilde{q}_{s\ell}^h}{p_s^\ell} \quad \forall s, \ell \in \mathbf{S} \times \mathbf{L}, \qquad (25)$$

We furthermore impose the *net position conditions*:

$$\mu_n^h . \tilde{\mu}_n^h = 0, \quad \forall n \in N, \qquad (26)$$

To understand these conditions, note that there is an indeterminacy in the players' actions, related to the money borrowed or deposited on short-term loans ($n \in \mathbf{S}^*$). Indeed, constraints (13) and (14), or (18) and (19), do not prevent an agent from being both a borrower and a lender in the short-term loan market s. Only her net situation is relevant in our liquidity constraints. Our condition means that agent h is either a borrower or a lender in each short-term loan s. There is no loss of generality in doing so: If an action satisfies our liquidity constraints, there exists an equivalent action that satisfies this additional, non-redundancy requirement. It is sufficient to redefine the actions as follows: $\tilde{\mu}_s^h := \max(\tilde{\mu}_s^h - \frac{\mu_s^h}{1+r_0}, 0)$ and $\mu_s^h = -(1+r_0)\min(\tilde{\mu}_s^h - \frac{\mu_s^h}{1+r_0}, 0)$. The hereby redefined action still satisfies the liquidity constraints and also our net position conditions.

When $n = \overline{0}$, on the contrary, the net situation is not equivalent to a position where an agent is on both side of the market, because he can default on the money he

[41] Such constraints are standard in strategic market games, cf. Giraud (2003).

[42] The product $\kappa^k \alpha_k^h$ is the quantity of goods stored by h as collateral for her sale of α_k^h units of asset k.

has borrowed. The liquidity constraints allow an agent to default on the borrowing side, while the Central Bank pays back her principal on the lending side. To avoid this odd situation, we therefore impose also our net position condition for long-term loans.

We therefore define the convex feasible set, Σ_η^h, as being the set of actions satisfying liquidity constraints (13), (14), (16), (18), (19), and (21), physical constraints (22) and (24), long-term collateral pledge (15), and net position conditions (26).

This choice yields utility $u^h(x^h)$ to player h.

3.5 Collateral Monetary equilibrium

Our definition extends the one introduced in Dubey and Geanakoplos (2003b) by adding the possibility of default both on financial assets and long-term loans.

We say that $\langle \eta, (\sigma^h)_{h\in\mathbf{H}}\rangle$ is a Collateral Monetary Equilibrium (CME) for the economy $\mathcal{E} := \langle (u^h, \mathbf{e}^h, m^h)_{h\in\mathbf{H}}, \mathbf{A}, M_0, M_{\overline{0}}, (M_s)_{s\in\mathbf{S}}\rangle$ if:

(i) All agents maximize:

$$\sigma^h \in \arg\max_{\tilde{\sigma}^h\in\Sigma_\eta^h} u^h\big(x^h(\eta,\tilde{\sigma}^h)\big)\ \forall h\in\mathbf{H}$$

(ii) All markets clear:

(a) Loans, $n \in N$:

$$\frac{1}{1+r_n}\sum_h \mu_n^h = M_n + \sum_h \tilde{\mu}_n^h \tag{27}$$

(b) Assets, $k \in \mathbf{K}$

$$\pi^k \sum_{h\in\mathbf{H}} \alpha_k^h = \sum_{h\in\mathbf{H}} \tilde{\alpha}_k^h. \tag{28}$$

(c) Commodities, $s\ell \in \mathbf{S}^* \times \mathbf{L}$

$$p_s^\ell \sum_h q_{s\ell}^h = \sum_h \tilde{q}_{s\ell}^h. \tag{29}$$

We will denote K_s the money that the central bank received in second period for the payment of the principal minus what she delivers to lenders:

$$K_s := \sum_h \Big[\min\Big\{p_s\cdot g_s^h(\kappa_{\overline{0}}^h)\ ;\ \frac{\mu_{\overline{0}}^h}{1+r_{\overline{0}}}\Big\} - \tilde{\mu}_{\overline{0}}^h\Big] \tag{30}$$

Note that $K_s \le M_{\overline{0}}$, with equality if no agent defaults on her long-term loan.

Remark that, when cast as a (type-symmetric) Nash equilibrium of the underlying strategic market game, this definition rests on the implicit assumption that players cannot condition their actions in period 1 on the actions observed from period 0.

This is consistent with the anonymity property of large markets.[43] Prices are the unique signal on which players coordinate.

Let us briefly comment on some specific aspects of the model that are responsible for its upshot.

a) As in most strategic market games, every transaction that an agent undertakes requires the physical transfer of money out of what he has on hand at that time. This amounts to various liquidity constraints. The upshot is that we have a well-defined physical process in which effect follows cause in a time sequence. By contrast, general equilibrium analysis steers clear of liquidity constraints because all transactions are imagined to occur simultaneously. The point of the present contribution is to go beyond this and to analyse the effects of liquidity constraints when default is permitted to occur on markets with collateralized assets. As we assume that each type of investor is represented by a continuum of negligible clones, they all take prices as given, which simplifies the analysis. The existence proof, however, provides the full-blown double auction underlying our model (see the Appendix).

b) We assume that agents may default on certain promises and not on others, and that the only consequence of default is forfeiture of collateral. For pawn shop loans, overnight repurchase agreements, margin loans and home mortgages, this assumptions is relatively close to reality. Repo loans, and mortgages in many states, are literally non-recourse loans. In the rest of the states, lenders rarely come after borrowers for more money beyond taking the house.

c) Money plays here all its different roles: it can be hold for transactional purposes (because of the liquidity constraints detailed *supra*) and as a store of value between periods 0 and 1. But it can also be used as an asset that permits transferring wealth from one state to another in period 1, hence as an insurance tool: if short-term interest rates are expected to be very high in some second-period state s, then agents will try to acquire money in advance in period 0. Furthermore, there may be also a speculative demand for money: inventorying money from period 0 to period 1 is equivalent to holding an implicit (risk-less, nominal) asset. If the return of this asset becomes more attractive, a speculative demand for it will appear. And finally, if commodity prices are expected to increase in the second period, there will be a demand for money on the long-term loan market driven by the fear for inflation.

It should be clear, however, that there is no money illusion: multiplying both outside and inside money by some constant λ solely amounts to computing prices, say, in cents rather than in euros. Since expectations are rational, the Central Bank's policy is also perfectly anticipated, so that the results to follow are not due to some irrational anticipations. And nevertheless, we shall see that the "stylized facts" evoked in the Introduction can be recovered within the present setting.

d) In the situation excluded by our two environments—that is, when there is no outside money and the long-run market is closed—then, every CME coincides with a barter collateral equilibrium as defined by Geanakoplos and Zame (2010). Indeed, then, interest rates will all be zero[44] while prices and inside money are homogeneous of degree 0, so that only relative prices are determined by the equilibrium conditions. As a consequence, the two environments precisely capture the ingredients that are

[43]See Giraud and Stahn (2003) for the impact of allowing for non-trivial monitoring in strategic market games with incomplete security markets.

[44]This will follow from Proposition 4.2.

necessary for the endogenous determination of the price level: Either there must be some outside money (in which case, the interplay of inside and outside money fixes absolute prices, as in Dubey and Geanakoplos (2003a)), either there is a collateral constraint associated with long-run loans (in which case, the interplay between this constraint and the quantity of money pumped in by the Bank on the long-run markets suffice to pin down nominal prices, as in Lin et al. (2010)).

4 General properties of Collateral Monetary Equilibria

Introducing collateral constraints in a model of incomplete markets has two well-known consequences. The standard non-arbitrage argument that lies at the core of pricing theory in the complete markets benchmark does no more hold, even *at equilibrium*, in our set-up where markets are endogenously incomplete due to the scarcity of collaterals. "Efficient financial markets" are usually said to be characterized by price processes that follow random walks. As is well-known, this martingale property is satisfied in GEI models (independently of the Pareto-inefficiency of its equilibria, see, e.g., Geanakoplos (1990)), but need no more be satisfied in our set-up with collateral requirements: when the collateral constraint is binding, its actual price is the sum of two shadow prices, the marginal value attributed to it by its marginal purchaser *plus* its value as a collateral (see, e.g., Cao (2010)). Hence, the market incompleteness induced by the collateralization of assets is of specific nature when compared to more classical models of market incompleteness. The second consequence is that equilibrium pricing is no more linear. Hence, the celebrated Modigliani-Miller theorem also fails in our setting as in any environment with non-linear pricing rules—which has long been recognized (Geanakoplos, 1990; Hellwig, 1982; Stiglitz, 1982).

4.1 Gains to trade

In this section, we adapt the intratemporal gains-to-trade assumption borrowed from Dubey and Geanakoplos (2003b) to our context with default.

Let $x^h \in \mathbb{R}_+^{\mathbf{S}^* \times \mathbf{L}}$ be any feasible allocation for household h. For any $\gamma \geq 0$, we say that $x = (x^h)_h \in \left(\mathbb{R}_+^{\mathbf{S}^* \times \mathbf{L}}\right)^{\mathbf{H}}$ is *not γ-Pareto optimal in state s* if there exist some trades $(\tau_s^h)_h \in \left(\mathbb{R}^{\mathbf{L}}\right)^{\mathbf{H}}$ in state $s \in \mathbf{S}^*$, such that

$$\sum_h \tau_s^h = 0 \qquad \text{(feasibility of trades)} \tag{31}$$

$$x_s^h + \tau_s^h \in \mathbb{R}_+^{\mathbf{L}} \quad \text{for all } h \in \mathbf{H} \quad \text{(consumability)} \tag{32}$$

$$u^h(\overline{x}^h[\gamma, \tau_s^h]) \geq u^h(x^h)\ \forall h \in \mathbf{H},\ \text{with at least one strict inequality} \quad \text{(improvement)} \tag{33}$$

where, for every $\ell \in \mathbf{L}$,

$$\overline{x}[\gamma, \tau^h]_{t\ell} := \begin{cases} x_{t\ell}^h & \text{if } t \in \mathbf{S}^* \setminus \{s\} \\ x_{s\ell}^h + \min\{\tau_{s\ell}^h, \frac{\tau_{s\ell}^h}{1+\gamma}\} & \text{for } t = s. \end{cases}$$

In words, the trades, τ^h, considered as candidates to γ-Pareto-improve x^h involve a tax of $\gamma/(1+\gamma)$ on trade. Of course, 0-Pareto-optimality coincides with the standard notion of Pareto-optimality. The *gains to trade*, $\gamma_s(x)$, in state s at a point $x \in \left(\mathbb{R}_+^{\mathbf{S}^*\times\mathbf{L}}\right)^{\mathbf{H}}$ is defined as the supremum of all γ for which x is not γ-Pareto-optimal in state s.

The stock of inside money injected in state s is M_s. The gains-to-trade hypothesis introduced in Dubey and Geanakoplos (2003b) compares the measure $\gamma_s(x)$ with a ratio of outside to inside money in state $s \in \mathbf{S}$ given by: $\hat{m}_s/M_s$. In our context, we need to slightly modify the definition given by Dubey and Geanakoplos (2003b) to $\hat{m}$ in order to account for the possibility of default. Defaulting, indeed, can be viewed as a way to endogenously create money at equilibrium, hence to expand the amount of "outside" money available to traders. At variance, however, with the standard no-default case, this amount of "outside money" becomes endogenous (since the interest rate charged on long-term loans is endogenous). Everything goes as if traders could not create more money by defaulting than their total credit extension on the long-term loan.[45] For our purposes, it therefore turns out that it will be sufficient to take:

$$\hat{m}_s := M_{\overline{0}} + \sum_h (m_0^h + m_s^h). \tag{34}$$

We will denote for $s \in \mathbf{S}^*$ $\overline{m}_s = \sum_h m_s^h$. Thus $\hat{m}_s = M_{\overline{0}} + \overline{m}_0 + \overline{m}_s$.

The gains-to-trade hypothesis can now be formulated as follows. For every state $s \in \mathbf{S}$, let us denote by X_s the subset of feasible consumption bundles that involve no trade in state s, that is:

$$\begin{aligned} X_s = \Big\{ &(x^h)_h \in \left(\mathbb{R}_+^{\mathbf{S}^*\times\mathbf{L}}\right)^{\mathbf{H}}, \exists (y^h)_h \in \left(\mathbb{R}_+^{\mathbf{L}}\right)^{\mathbf{H}}; \\ &\forall t \in \mathbf{S}, \sum_h x_0^h + \sum_h y_0^h = \sum_h e_0^h, \sum_h x_t^h = \sum_h e_t^h + \sum_h g_t^h(y^h); \\ &\forall h \in \mathbf{H}, x_s^h = e_s^h + g_s^h(y^h) \Big\} \end{aligned} \tag{35}$$

The first condition implies that the allocation $(x^h)_h$ is feasible (involving possibly trades among players and intertemporal storage), the second condition that it involves no trade in state s.

Gains-to-trade hypothesis. For all $s \in \mathbf{S}$ and every $x \in X_s$, $\gamma_s(x) > \frac{\hat{m}_s}{M_s}$.

This assumption requires that there be gains to trade in *every* state $s \in \mathbf{S}$ in period 1, but not necessarily in period 0. It also rules out the case of only one commodity per state, because then, any feasible and consumable allocation would be automatically 0-Pareto optimal. Similarly, it rules out the representative agent case where $H = 1$, because, again, this would lead to Pareto-optimality for free.

If initial endowments in the economy $\mathcal{E}$ are not Pareto-optimal, then as $M_s \to +\infty$ leaving the economy otherwise fixed, the Gains-to-trade hypothesis will sooner or later be satisfied.

[45]See Step 6 in the proof of Theorem 1.

4.2 Existence

The next result (whose proof is in the Appendix A.1) extends the existence theorem of Dubey and Geanakoplos (2003b) to the case where households can default on the delivery of collateralized derivatives or on their long-term loans. Alternatively, it extends the existence result of Geanakoplos and Zame (2010) by introducing money in their incomplete markets economy with collateralized assets. Similarly, it extends Lin et al. (2010) to the case where investors can default not only on the Bank's long-term loans but also on their financial assets. The three key ingredients of our approach are: a) the introduction of our HDD hypothesis; b) the combination of both an endogenous collateral constraint for long-term loans and an exogenous collateral constraint for financial assets ; c) the adaptation of the gains-to-trade assumption introduced by Dubey and Geanakoplos (2003a) in order to cope with defaultable assets and loans ; d) that some money be injected (otherwise monetary equilibria trivially cease to exist) : $\overline{m}_0 + M_{\overline{0}} > 0$. Notice, therefore, that we do not impose that money be injected in the short-run, provided the long-run monetary market is active.

We prove existence under two distinct environments. In both frames, $\overline{m}_0 + M_{\overline{0}} > 0$ is assumed. Now, either A) aggregate outside money is positive in period 0 ($\overline{m}_0 > 0$); or B) $\overline{m}_0 = 0$ but $M_{\overline{0}} > 0$. Notice that, in environment A, the Central Bank may refuse to intervene on the long-term loan market (i.e., a CME exists even when $M_{\overline{0}} = 0$)—in this case, however, it has to pump in a positive stock of inside money in the long-term market. Moreover, in both environments, an equilibrium exists even absent of any financial asset. The situation ruled out by environments A and B is the one where $\overline{m}_0 = M_0 = 0$.

Theorem 1 *Under environment A or B, any monetary economy $\mathcal{E}$ verifying our standing assumptions together with the gains-to-trade hypothesis has a* CME.

Existence holds therefore for a broad class of economies involving real and/or nominal assets, options, derivatives, and even more complicated non-linear assets. In the standard framework with no money and no collateral constraints, the presence of such assets implies that the space of feasible income transfers does not depend continuously on commodity prices, so that equilibrium may not exist.[46] In the present framework however, there are two forces which help restore existence: both the collateral requirements and the cash-in-advance constraints place an endogenous bound on short-sales. The first one because of the scarcity of collateralized assets (see, e.g., Geanakoplos and Zame (2002)), the second, because of the scarcity of money (see, e.g., Dubey and Geanakoplos (2003b)). As in the standard incomplete markets setup (see Radner (1972) for instance), a lower-bound on short-sales eliminates the discontinuity and guarantees existence.

It should be stressed that no Gains-to-trade hypothesis is needed in period 0 in order to guarantee the existence of an active Collateral Monetary Equilibrium. Nevertheless, it should be clear that the Pareto-optimality of any period 0 equilibrium allocation depends upon r_0 and $r_{\overline{0}}$. The smaller are these interest rates, the closer will be the allocation to optimality. As a consequence, monetary authorities may be willing to increase M_0 and $M_{\overline{0}}$ in order to improve the optimality of trades. The

[46] See Duffie and Shafer (1985) for a generic existence proof, Ku and Polemarchakis (1990) for a robust example of non-existence with options.

next section is devoted to the implications of such a monetary policy. As a matter of preparation, we need to understand a couple of properties of the equilibrium yield curve.

4.3 The yield curve

It is easy to show that money, in our model, is non-neutral (Dubey and Geanakoplos, 2003a,b). Nevertheless, we get the analogue of a quantity theory of money:

Proposition 4.1 *(i) At equilibrium, one has:*

$$\sum_h \pi^k \alpha_k^h + \sum_h p_0 \cdot q_0^h \leq \sum_h m_0^h + M_0 + M_{\overline{0}}. \tag{36}$$

With equality, as soon as $r_0 > 0$.
(ii) For every state $s \in \mathbf{S}$, *one has*

$$\sum_h p_s \cdot q_s^h \leq \sum_h m_s^h + \sum_h m_0^h + M_s + M_{\overline{0}} - r_0 M_0 - r_{\overline{0}} M_{\overline{0}} \tag{37}$$

and[47]

$$\sum_h p_s \cdot q_s^h \leq (1 + r_s) M_s + K_s \tag{38}$$

With equality, as soon as $r_s > 0$.

Proof. For (i), the liquidity constraint (14) is binding. For (ii), (19) is binding as well as (21). See the Appendix A.2 for a detailed proof. □

The next Proposition describes the term structure of interest rate, showing that the full interplay of all the demands for money can be captured in our model (transaction, precaution, speculation, storage, insurance against inflation).

Proposition 4.2 *At any* CME,
(i) $r_s \geq 0 \ \forall s \in \mathbf{S}^*$;
(ii) $r_0 M_0 + r_{\overline{0}} M_{\overline{0}} + r_s M_s \geq \sum_h \big(m_0^h + m_s^h\big) \ \forall s \in \mathbf{S}$ *with an equality if, and only if, there is no default on the long-run money market,* $\overline{0}$, *in state s;*
(iii) Suppose utilities are separable, only second-period consumption matters, $u^h\big(x_0^h, x_1^h\big) =: v^h\big(x_1^h\big)$, *each storage mapping reduces to the identity map:* $g_s^h(x) = x \ \forall x, h$, *and all goods are eligible as collateral* $\mathbf{C} = \mathbf{L}$. *Then* $r_{\overline{0}} \geq \min_{s \in \mathbf{S}} 1 + r_0 - 1/(1 + r_s)$ *with strict inequality unless all* r_s *are identical* $\forall s \in \mathbf{S}$;
(iv) $r_0 \leq ((1 - r_{\overline{0}}) M_{\overline{0}} + \sum_h m_0^h)/M_0 =: \mu_0(m, M)$ *and* $r_s \leq \mu_s(m, M) \ \forall s \in \mathbf{S}$, *where* $\mu_s(m, M)$ *is the ratio of outside to inside money in state s:* $\mu_s(m, M) := \frac{\hat{m}_s}{M_s}$, *with*

$$\hat{m}_s := \sum_h (m_0^h + m_s^h) + M_{\overline{0}}.$$

(v) $r_{\overline{0}} < 1 + r_0$.

[47] Recall that K_s is the money received by the central bank on long-term loans, see (30).

Remark. Observe that, under both environments considered in subsection 4.2. supra, $\hat{m}_s > 0$ for every s (although we do not impose that $\overline{m}_s = \sum_h m_s^h > 0$).

Proof.

(i) If the contrary, a player could improve her profile by borrowing more money, spending a little on commodity, inventoried the money to pay back the extra-loan.

(ii) No agent has some money on hand at the end of period s (for she would rather spend it or borrow more to spend), so $\Delta(21) = 0$. This yields

$$r_0 M_0 + r_s M_s + r_{\overline{0}} M_{\overline{0}} = \sum_h m_s^h + \sum_h m_0^h + M_{\overline{0}} - K_s \tag{39}$$

The result follows because $K_s \leq M_{\overline{0}}$ with equality if and only if there is no-default. Note that money created by default on long-term loan is like outside money.

(iii) We have $\frac{\sum_h \mu_0^h}{1+r_0} = M_0 + \sum_h \tilde{\mu}_0^h$. Because $M_0 > 0$, at least one player h is a net borrower on M_0. Suppose the claim is false. Let h borrows ϵ less on M_0 but more on $M_{\overline{0}}$. Player h can still act in period 0 as before: she buys the same goods with the $\overline{0}$ money as with the 0 money (because $\mathbf{C} = \mathbf{L}$ she can pledge them as collateral), and she has the same utility (because only second period consumption matters and storage map are identity). Because she has less to pay on her loan 0, she inventories $\epsilon(1 + r_0 - r_{\overline{0}})$ of money into period 1. She deposits only $\epsilon/(1 + r_s)$ on M_s. This will exactly reimburse the principal of her long-loan. She is endowed with the extra-money $\Delta((1 + r_0 - r_{\overline{0}}) - 1/(1 + r_s))$ (which is positive in every state s *ad absurdum*) that she can spend to increase her utility. A contradiction.

If the r_s^ε are not identical and if there is equality in the formula, then the extra-money in every state s is non-negative but positive for some state, and the agent can still increase her utility.

(iv) Summing over h liquidity constraints from (13) to (16), one gets: $r_0 M_0 \leq (1 - r_{\overline{0}}) M_{\overline{0}} + \sum_h m_0^h$. For inequality on r_s, see step 6 of the proof of Theorem 1 in Appendix A.1.

(v) Suppose that $r_{\overline{0}} \geq 1 + r_0$. Instead of borrowing long-term, the agent h can borrow short-term, buy the same goods as before and enjoys them from period 0, pay-back the loan with the money she would have spent on the long-term loan interest. This increases her utility. A contradiction.

□

Corollary 4.1 *(i) Suppose that M_0 (resp. $M_{\overline{0}}$) grows to infinity. Then, along any sequence of corresponding* CME*, $r_0 \to 0^+$ (resp. $r_{\overline{0}} \to 0^+$).*

(ii) Suppose that $M_n \to \infty$, with $n \in \{0, \overline{0}, s \in \mathbf{S}\}$. Then, if r_n remains bounded below above 0*, some default must occur on the long-run market,* $\overline{0}$.

Proof.

(i) is a simple consequence of (37). (ii) follows from Proposition (4.2) (ii).

□

4.4 Fiscal deficit and bankruptcy of the Central Bank

When outside money is interpreted in terms of public expenditure, property (ii) has an important implication in terms of fiscal deficit.[48] On the left hand of the inequality

[48] As already said, several alternative interpretations of outside money are conceivable: When viewed as cash inherited from some unmodelled past default, the conclusions of this subsection fail.

(ii), there is the interest revenue of the Bank, and on the right, the Treasury's expenditures. This equation thus says that the Treasury is balancing its budget on the long-run as long as there is no default, although M_s and m_s^h may be quite arbitrary: seigniorage (i.e., $r_0 M_0 + r_{\overline{0}} M_{\overline{0}} + r_s M_s$) covers exactly the cost of public expenditures. On the other hand, whenever there are defaults on the long loan market, in some state s, the public deficit is given by:

$$\text{Public deficit in state } s := \sum_h (m_0^h + m_s^h) - r_0 M_0 - r_{\overline{0}} M_{\overline{0}} - r_s M_s = K_s - M_{\overline{0}}.$$

Whenever there is no outside money in the economy (i.e., $\overline{m}_s = 0\ \forall s \in \mathbf{S}^*$), then (ii) implies that a positive long-run interest rate, $r_{\overline{0}} > 0$ is only possible, at equilibrium, when there is some default on the long-run market. In the latter case, the Central Bank loses the difference, $M_{\overline{0}} - K_s$, between the money it lends and the cash it finally recovers. If this difference exceeds the Central Bank's equity, this means that our equilibrium concept is quite compatible with the Bank being bankrupt. Theorem 2 below will even show that, under certain circumstances (scenario (iib)), the Bank may lose its total claim (i.e., $K_s = 0$) —a situation in which the Treasury would have to recapitalize it. To take but an example, within its LTRO programme, the ECB lent about €1 trillion between December 2011 and February 2012 over 3 years at 1% rate, while its equity amounts to €80 billion.

4.5 Bid-ask spread

The next proposition enables to define endogenous bid-ask spreads, depending upon the interest rates, $\big(r_0,\ (r_s)_s\big)$. Consider some pair, $\langle \eta, \sigma^h \rangle$, of macrovariable and strategy profile. For each financial asset k, denote the corresponding final portfolio of agent h by

$$\theta_k^h := \frac{\tilde{\alpha}_k^h}{\pi_k} - \alpha_k^h \tag{40}$$

Also, denote the net trade of agent h on commodity ℓ in state $s \in \mathbf{S}^*$ by

$$z_{s\ell}^h := \frac{\tilde{q}_{s\ell}^h}{p_s^\ell} - q_{s\ell}^h \tag{41}$$

Equation (42) below says that, in the first period, the buying price of commodity ℓ (resp. asset k) is p_0^ℓ (resp. π^k), while its selling price is $p_0^\ell/(1+r_0)$ (resp. $\pi^k/(1+r_0)$). The ratio $r_0/(1+r_0)$ can therefore be interpreted as a bid-ask spread. Similarly, the spread on the commodity market in state s is $r_s/(1+r_s)$.

Proposition 4.3 *Let $\langle \eta, \sigma^h \rangle$ be a macrovariable and a strategy of player h such that $\sigma^h \in \Sigma_\eta^h$ for every h. Then,*

$$p_0 \cdot z_0^{h+} + \pi \cdot \theta^{h+} - \frac{1}{1+r_0}\left(p_0 \cdot z_0^{h-} + \pi \cdot \theta^{h-}\right) \leq m_0^h + \frac{\mu_{\overline{0}}^h}{1+r_{\overline{0}}}\left(1 - \frac{r_{\overline{0}}}{1+r_0}\right), \tag{42}$$

and

$$p_s \cdot z_s^{h+} - \frac{1}{1+r_s} p_s \cdot z_s^{h-} \leq \Delta(17) + m_s^h - \frac{\gamma_s^h}{1+r_s}, \tag{43}$$

where $\Delta(17) + m^h_s$ *is the money in the hand of agent* h *at the beginning of period* 1 *and* $\gamma^h_s := \min\{p_s \cdot g^h_s(\kappa^h_{\overline{0}}); \frac{\mu^h_{\overline{0}}}{1+r_{\overline{0}}}\} - \tilde{\mu}^h_{\overline{0}} + \sum_k \mathbf{A}^k_s(\eta_0, \eta_s)\left(\alpha^h_k - \frac{\tilde{\alpha}^h_k}{\pi^k}\right)$ *denotes the net money paid back by agent* h *on the long-loan market and on securities.*

Proof. Note first that $z^{h+}_{s\ell} - \frac{z^{h-}_{s\ell}}{1+r_s} \leq \frac{\tilde{q}^h_{s\ell}}{p^\ell_s} - \frac{q^h_{s\ell}}{1+r_s}$ and similarly $\theta^{h+}_k - \frac{\theta^{h-}_k}{1+r_0} \leq \frac{\tilde{\alpha}^h_k}{\pi_k} - \frac{\alpha^h_k}{1+r_0}$. We have therefore

$$\begin{array}{ll} & p_0 \cdot z^{h+}_0 + \pi \cdot \theta^{h+} - \frac{1}{1+r_0}\left(p_0 \cdot z^{h-}_0 + \pi \cdot \theta^{h-}\right) \\ \leq & \sum_\ell \tilde{q}^h_{0\ell} + \sum_k \tilde{\alpha}^h_k - \frac{1}{1+r_0}\left(p_0 \cdot q^h_0 + \pi \cdot \alpha^h\right) \\ \leq & m^h_0 + \frac{\mu^h_{\overline{0}}}{1+r_{\overline{0}}}(1 - \frac{r_{\overline{0}}}{1+r_0}) - \tilde{\mu}^h_{\overline{0}}(1 - \frac{r_{\overline{0}}}{1+r_0}) \end{array}$$

For the last inequality we have subtracted to the liquidity constraint (14) the constraint (16) divided by $1 + r_0$ and used the fact that $\mu^h_0 = 0$ and $\Delta(14) = 0$. Since $1 + r_0 > r_{\overline{0}}$, $\tilde{\mu}^h_{\overline{0}}(r_{\overline{0}}/(1 + r_0) - 1) \leq 0$, the inequality follows.

To prove the inequality in second period, one proceeds in the same way, noting that (19) is binding. □

5 Robust liquidity trap versus financial crash

As shown by the Gains-to-trade hypothesis, the monetary authority will be able to improve the efficiency of trade, and thus total real output, by increasing supplies of Bank money or, equivalently, by lowering interest rates. We now show that, when doing so, monetary authorities face a universal dilemma. This dilemma, when stated in full generality, says that three, and only three, scenarios are compatible with the equilibrium condition: either the economy falls into a liquidity trap in period 0, or the Central Bank circumvents the trap but at the cost of fuelling domestic inflation, else the monetary policy encourages impatient or optimistic agents in accumulating debts in period 0, but at the cost that, with positive probability, deleveraging in the second period forces agents to cut spending, which results in a collapse of trades and default on the long-run market—actually, a complete loss of its loans by the Central Bank.

Define the monetary policy of the Central Bank as involving a collection $\mathcal{M} := \langle \mathcal{M}_s(\cdot)\rangle_{s\in\mathbf{S}}$, of mappings taking value in $\mathbb{R}_+$, where, for each s, $\mathcal{M}_s : M_0 \mapsto M_s$. The collection, $(M_{\overline{0}}, \mathcal{M})$, defines the Central Bank's monetary policy and is publicly announced before markets open in the first period. Given a monetary policy, under the assumptions of Theorem 1, there exists a CME associated with any choice of M_0. In the next result, we fix a monetary policy and consider the impact of varying M_0. We call a *spot liquidity trap in state* s a CME where the short-term interest rate in s hits the zero lower-bound, $r_s = 0$ (but $r_{s'}$ might differ from 0 for $s' \neq s$).

In the sequel, we call *liquidity trap in state* $s \in \mathbf{S}^*$ the following situation: there exists some M^*_0 such that $\forall M_0 \geq M^*_0$, $r_0(M_0) = 0$, (resp. $r_s(M_0) = 0$) and households horde at least $M_0 - M^*_0$ (resp. $\left[M_s(M_0) - M_s(M^*_0)\right]^+$) as unspent money balances in period 0 (resp. state s) and prices remain constant, $p_0(M_0) = p_0(M^*_0)$ (resp. $p_s(M_0) = p_s(M^*_0)$).

Theorem 2 *Suppose that the assumptions of Theorem 1 are in force. Fix a monetary policy,* $(M_{\overline{0}}, \mathcal{M})$ *and let* $M_0 \to +\infty$*. One of the three following situations must arise at equilibrium:*

(i) A liquidity trap occurs in some state $s \in \mathbf{S}^*$*;*

(ii) or $r_0(M_0) > 0$ *and* $r_s > 0$*, for every* $s \in \mathbf{S}$*, as* $M_0 \to +\infty$*. Then*

Either (iia) commodity prices explode to infinity in all states $s \in \mathbf{S}^*$ *(i.e.,* $\|p_0^n\|$ *and* $\|p_s^n\| \to +\infty$*) —this is the "inflationary scenario".*

Or (iib) $\sum_h q_s^h(M_0) \to 0$ *as* $M_0 \to +\infty$*, i.e., trades vanish in some state* s*. This is the "collapse scenario". Moreover, when there is no inflation, this collapse of trades is accompanied by a complete loss of the Central Bank on the long-run monetary market (i.e.,* $K_s = 0$[49]*). This is the "crash variant" of the collapse scenario.*

Proof. Take some $(M_{\overline{0}}, \mathcal{M})$ as given. A CME exists for every choice of M_0. Suppose the monetary authority increases M_0, and keeps $(M_{\overline{0}}, \mathcal{M})$ fixed. For every given M_0, a finite amount of inside money, $M_s := \mathcal{M}_s(M_0)$, is injected at time 1. Thus, the total stock of money available to be spent in state s is no more than $M_{\overline{0}} + M_s(M_0) + m_0 + m_s$.

Three cases are in order, which are not mutually exclusive and partially overlap with the three scenarios of the Theorem. At the end, however, each case must lead to one of the three scenarios.

(α) A liquidity trap emerges in state $s \in \mathbf{S}^*$: there is some M_0^* such that $M_0 \geq M_0^*$ implies $r_0 = 0$ or $r_s = 0$ for some s. The hoarding of real money balances thereafter increases proportionately with $M_0 - M_0^*$. The economy has therefore reached a liquidity trap when $M_0 \geq M_0^*$. This first case coincides with scenario (i).

(β) Suppose that $r_s > 0$ for every $s \in \mathbf{S}^*$. Assume, furthermore, that the quantity $M_s = \mathcal{M}_s(M_0)$ increases to infinity with M_0 for *every* $s \in \mathbf{S}$. Note that, because of Proposition (4.2) (iv), second period spot interest rates $r_s \to 0^+$ as $M_0 \to +\infty$, for every $s \in \mathbf{S}$. Will this expansionary scenario inevitably lead to inflation in both periods? Inflation either on commodities or assets (possibly both) in period 0 follows from $r_0 > 0$, (36) and the fact that transactions are uniformly bounded—either because of the physical constraints on commodities or because of the collateral requirements on derivatives (22) together with the scarcity of collaterals. Regarding state s, if M_s grows to infinity for every s, everything else being kept fixed, then the CME allocation will eventually converge to some (barter) collateral equilibrium.[50] Indeed, if follows from Proposition (4.2)(iv) that $r_s \to 0^+$ while all cash balances (money in the right-hand-side of (18) as well as debts) vanish. Therefore, since at the limit "real transactions" will be nearly constant (and equal to the asymptotic barter collateral equilibrium), the further increase of inside money will induce an increase of the "price level" as a consequence of (36) and (38) (recall that $r_s > 0$).

This is the "inflationary scenario" (iia).

(γ) Assume that we are neither in case (α), so $r_0 > 0$ and $r_s > 0$ for every s, nor in case (β): in some state s, M_s is uniformly bounded for M_0.

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[49] Recall that K_s is defined by (30). Of course, whenever the long-run monetary market is closed, $K_s = 0$ in every scenario.

[50] See, e.g., Geanakoplos and Zame (2010).

Suppose that the volume, $\sum_h q_s^h(M_0)$, of equilibrium aggregate supply of spot commodities in this state is bounded from below by some (L-dimensional) lower-bound, $\underline{q}$, independent from M_0. This means that spot prices, p_s, in state s, must have an upper-bound independent from M_0, as follows from (19) and the boundedness of M_s. As a consequence, commodity and asset prices at time 0 must be bounded by some constant, say, K, independent of M_0. Otherwise, indeed, every agent h could sell $\varepsilon > 0$ of any commodity she is positively endowed with at time 0, and buy the whole economy for state s of period 1. Similarly, security prices must be uniformly bounded, independently of M_0: Indeed, given our assumption that $\sum_h \mathbf{e}_{0\ell}^h > 0$, for each security there exists at least one household which is furnished in the commodities that are eligible as collateral for j, and which could otherwise go short in this security in order to buy the whole economy in the next period.[51]
The inside money borrowed on loan 0 must be spent in time 0 : no agent would borrow money on loan 0 at positive interest unless she is going to spend it in that very period. But the boundedness of prices and of quantities (because of physical constraints) would then contradict the quantity theory equation (36).

So,

$$\lim_{M_0 \to +\infty} \sum_h q_s^h(M_0) = 0. \tag{44}$$

This is the "collapse scenario" (iib). If $M_s(M_0)$ is bounded away from zero (i.e., $\exists \eta > 0,\ M_s(M_0) \geq \eta\ \forall M_0$), equality (38) then implies that $\|p_s(M_0)\| \to +\infty$, and we are also in the inflationary scenario (iia). If there is no inflation in state s (for this it is necessary that $\lim_{M_0 \to \infty} M_s(M_0) = 0$), then (38) implies that $K_s = 0$. This is the "crash variant".

□

The next corollary shows that unconventional monetary policy (that consists in intervening on the long-run monetary market) does not help the Central Bank circumvent the monetary dilemma. Suppose, therefore, that $(M_0, (\mathcal{M}_s)_{s\in\mathbf{S}})$ are fixed, and let the Bank choose $M_{\overline{0}}$.

Corollary 5.1 *Under the assumptions of Theorem 2, suppose that $M_{\overline{0}} \to +\infty$, while $(M_0, (\mathcal{M}_s)_{s\in\mathbf{S}})$ is kept fixed. The same conclusion as in Theorem 2 folllows.*

Proof.

The proof follows *verbatim* that of Theorem 2. The contradiction induced by the Quantity Theory equation (4.1) arises in the same way when $M_{\overline{0}} \to +\infty$ as when $M_0 \to +\infty$. □

The strength of Theorem 2 and Corollary (5.1) is to show that there is actually no escape road from what we have called the monetary dilemma : Either the Central Bank commits to fostering inflation, or it takes the risk of either a liquidity trap or a collapse of trades due to a crash on the long-run monetary market. Scenarios (i) and (ii) are obviously mutually exclusive so that, if neither (i) nor (iia) occur, then (iib) must hold. What this dilemma does *not* prove is that *each* of these three regimes may actually take place. This is why it needs to be supplemented by the Example of section 2. The dilemma does also not claim that the three scenarios are mutually

[51]See steps 3 and 4 of the proof of the existence Theorem 1 in the Appendix A.1 for same arguments in details.

exclusive: one may have inflation in spot commodity prices of state s together with a liquidity trap in period 0. Similarly, a liquidity trap in state s is compatible with inflation in period 0.

Remark 1. In case a liquidity trap occurs in the first period, as the Bank increases M_0, holding $M_{\overline{0}}$ fixed, the short-term nominal interest rate, r_0, will eventually hit the zero lower-bound while M_0 is still finite. Further increases of M_0 will have no effect on prices or on trades, but simply induce the households to hold larger real money balances. Households will borrow the extra money at zero cost, hoard it in their pockets, and then return it unused at the end of period 0. At this level of generality, the emergence of a liquidity trap does not have any welfare implication. However, if a liquidity trap occurs in environment A, the resulting allocation must be Pareto-suboptimal. Indeed, in environment A, outside money is non-zero, so that Proposition (4.2) (ii) implies $r_s > 0$ for some $s \in \mathbf{S}^*$, so that trades must be inefficient. Observe that this "liquidity trap" is robust to any (infinitesimal) perturbations of the fundamentals of the economy. Therefore, it should not be confused with the phenomenon exhibited in Dubey and Geanakoplos (2003b), where, absent collateral constraints, it is shown that a liquidity trap emerges as soon as the underlying barter GEI economy (i.e., the incomplete markets economy where $M_s = +\infty$ for each $s \in \mathbf{S}^*$) has no competitive equilibrium. Indeed, it is known from Duffie and Shafer (1985) that, generically, this barter GEI economy admits an equilibrium, which implies that the liquidity trap of Dubey and Geanakoplos (2003b) is non-generic. Here, by contrast, it may be robust, as shown by the example of section 2.

The next corollary provides a sufficient condition for a first-period liquidity trap to occur.

Corollary 5.2 *Under the assumptions of Theorem 2, suppose $M_0 \to +\infty$. If trades in a state $s \in \mathbf{S}$ are bounded away from 0 ($\sum_h q_s^h(M_0) \geq \varepsilon$, for some $\varepsilon > 0$ independent from M_0) and $M_s(M_0)$ is bounded, then a first-period liquidity trap* must *occur.*

Proof.

The argument also closely follows the proof of Theorem (2): since a M_s is bounded, and trades don't vanish in the second period, second-period prices must be bounded. Since we are at equilibrium, first-period prices must be upper-bounded as well. (4.1) then implies that there exists some M_0^* for which $r_0(M_0) = 0 \; \forall M_0 \geq M_0^*$.
□

The next Corollary offers a different look at the monetary dilemma, starting from the various options that are available to the central banker. Its proof follows immediately from that of Theorem 2) and Corollary (5.1).

Corollary 5.3 *Within the set-up of Theorem 2, suppose that either M_0 or $M_{\overline{0}} \to +\infty$, and the economy does not reach a liquidity trap.*

(i) If $M_s(\cdot) \to \infty$, the level of prices in state s grows to infinity in state $s \in \mathbf{S}$.

(ii) If the Central Bank wants to avoid inflation in state s, then it must decide for a harsh monetary contraction, namely $M_s(\cdot) \to 0^+$. But at the cost of being sure to lead the economy to the crash variant of scenario (iib) of Theorem 2, i.e., to a collapse of trades and a crash on the long-run market.

(iii) Finally, if $M_s(\cdot)$ is caped and floored above 0 *(independently of M_0 and $M_{\overline{0}}$), both scenarios will occur: an unlimited growth of prices (as either M_0 or $M_{\overline{0}}$ increases) together with a collapse of trades.*

Remark 2. The inflationary scenario (iia) means that, when the amount of injected inside money lies above a certain threshold (which depends upon the characteristics of the economy), then, the classical dichotomy holds and inflation is the sole output of further monetary injection. Below this threshold, however, an increase of injected money has an ambiguous impact.[52] This leaves some room for an "optimal corridor" where an expansionary monetary policy is sufficiently credible to avoid scenario (i) but not unduly "laxist", so as to avoid unnecessary inflation.

Remark 3. Suppose that, in each second period-state, there exists at least one consumer who derives no utility from (part of) her initial endowment. At equilibrium, she will put this part for sale, so that $\sum_h q_s^h(M_0)$ will be bounded from below in each state s. In this peculiar case, the crash scenario cannot occur, and we are only left with scenarios (i) and (iia).

6 Some concluding comments

Let us close this paper with a few comments.

A) This paper does not attempt to provide a unified analysis of the global crisis that started in 2007. Nevertheless, the model presented above, and its main results, shed some light about what we may have learned from the crisis and the policy issues raised by the response of the authorities to it. The monetary dilemma highlighted by Theorem (2) says that there are three, and only three, scenarios compatible with the Nash equilibrium conditions:

- scenario (iia): The size of injected money, $(M_0, (M_s)_s)$ allows to improve the efficiency of trades at the cost of a possibly unbounded inflation of commodity and asset prices in both periods.

- scenario (i): Inflation is prevented in the first period but at the cost of a liquidity trap.

- scenario (iib): rational exuberance on financial markets leads to a global crash in the second period.

The introduction of collateral requirements into monetary general equilibrium analysis enables to emphasize the role of leveraging as one of the microeconomic roots of financial crashes. Indeed, as shown by section 3 above, the larger the leverage ratio, the larger are the debts of optimistic investors in case of default. It has been argued, e.g., by Adrian and Shin (2008) that, even in the absence of a true bankruptcy, the very fact that a bank's assets have lost value implies a sudden rise in the leverage ratio, which is likely to lead the bank to sell off assets or restrict credit in order to deleverage.[53] This, however, is a partial equilibrium argument. Here, we can recast the argument within a general equilibrium framework: it is the shift of wealth

[52] This confirms the remark already made for one-shot economies by Dubey and Geanakoplos (2003a).

[53] Large European banks in 2007 had leverage ratios between 20 in the UK and 35 in Switzerland (see Panetta et al. (2009)).

between optimistic investors and pessimistic ones that can create a dramatic fall in prices and, eventually, a crash.[54]

This is not to say that our story depicts financial crashes as "black swans", i.e., as large-impact, low-probability events against which any protection would be exceedingly costly.[55] According to our model, there are two ways to circumvent the risk of a big crash. The first one consists in turning to a contractionnary monetary policy—at the cost of running the risk of falling into the liquidity trap (i.e., of shifting from scenario (iib) to scenario (i)). The second way consists in regulating financial markets so as to reduce their leverage ratio (driven by β in the Example of section 2)—at the cost of having to accept a high inflation rate on consumer prices whenever the monetary policy turns out to be too expansionary.

B) The interplay between money and collaterals enables to show that monetary policy and financial markets are deeply connected. At least since 2008, it has been argued that the scope of the Central Bank's supervision of inflation should be enlarged so as to include inflation of financial assets. This debate can be recast within our general equilibrium set-up with rational expectations: Indeed, the quantity M_0 of money injected on the short-term loans market may fuel an inflation of asset prices when leverage on the financial markets is sufficiently large. This means that one explanation of the Great Moderation (and of the fact that consumer price inflation remained rather subdued throughout the 2000–06 period) might rest on the sharp increase of leverage ratios on financial markets. Despite the vivid growth of the world monetary basis (15% each year since 1997, 30% since 2007) we did not observe the domestic inflation we should have experienced according to a naive interpretation of the Quantity Theory of Money (equation (36)) because this huge amount of fresh liquidity migrated from the real sector to the financial sphere.

On the other hand, however, Theorem 2 shows that the deflation risk is perfectly compatible with rational expectations and market clearing. Thus, when then-board member Ben Bernanke famously outlined a contingency plan to avoid the repetition of the Japanese experience (Bernanke (2002)), our model suggests that he was not referring to some improbable curiosity: the liquidity trap is part of an equilibrium story with rational expectations. Moreover, our dilemma shows that, whenever a Central Bank efficiently accomplishes its mission dedicated to consumer-price stability (i.e., avoids scenario (iia)), then it faces only two alternative scenarios: Either inflation on financial markets driven by some "rational exuberance" whose burst may induce a global collapse (scenario (iib)) or a liquidity trap (scenario (i)). In scenario (iib), if the Central Bank sticks to consumer-price stability it will have little reason to raise interest rates aggressively, and will therefore be unable to fight against rational exuberance, hence, to prevent a crash. Thus, our approach provides a theoretical ground for a plea in favor of Central Banks standing ready to depart from their price stability goal in the name of financial stability.

In scenario (i), application of the Taylor benchmark encounters the zero-bound problem: While the Taylor rule would recommend a negative interest rate, this is impossible to achieve because rational depositors are not prepared to pay for keeping deposits.[56]

[54] See Geanakoplos (2001) for a seminal statement of this phenomenon, ks03, Kiyotaki and Moore (1997), and Geanakoplos and Zame (2002), for further work.

[55] For a defense of the "Black Swan" viewpoint, see Blanckfein (2009).

[56] We did not formally define the Taylor rule in our present framework. There are two reasons for

Thus, our approach also makes the case for unconventional monetary policies in order to avoid liquidity traps. The recommended policy, however, is a striking variant of the zero-interest rate policy (ZIRP). Imagine, indeed, that the Central Bank prints vast amounts of banknotes and drops them from helicopters. Individuals receiving banknotes from heaven could feel suddenly richer and could spend at least part of this money, especially if they have heard about monetarism and fear that relying on the printing press will in the end induce inflation. Demand should pick up and inflation would indeed follow later on. As we have seen, this reasoning, however, does not necessarily hold here: the quantity theory of money, in the present model, does not imply the classical dichotomy for any level of printed money.[57] Thus, economic agents know that the Central Bank's power to create money does not automatically result in inflation. If, indeed, investors are convinced that, in the second period, the Central Bank won't pursue its easing policy, the scenario (i) of our narrative tells us that they will horde the helicopter money unspent. For this additional money to help the economy escape from the liquidity trap, the Bank must convince the economic agents that it will go further in its zero-interest rate policy, hence should commit to put zero interest rates in the second period as well: we are then back to our scenario (iia).

The issue at hand therefore becomes to find channels by which the Central Bank can commit, implicitly or explicitly, to higher inflation in the future (i.e., to even lower rates and more liquidity in the second period). Thus, our approach sustains the viewpoint vividly expressed by Krugman (2000) (and later by Orphanides (2004)) in the context of the Japanese crisis: the Central Bank of Japan "needs a credible commitment to expand not only the current but also *future* money supplies, which therefore raises future expected prices—or, equivalently, a credible commitment to future inflation" Krugman (2000), Theorem 2 shows that, *there is no alternative* to such an "irresponsible" commitment, as otherwise the Central Bank faces two major failures —either the deflationary liquidity trap or the possibility of a financial crash. This absence of a fourth scenario (where the Central Bank could avoid any disaster and still commit to be "responsible"' with respect to consumer prices) is what we have called the "dilemma of non-conventional monetary policy". How can the Central Bank proceed to such a commitment? For instance, by monitoring long-term rates, $r_{\overline{0}}$. Central Banks normally only target the short end of the yield curve, leaving the determination of longer-term interest rates to market mechanisms. In a situation of near-deflation, however, the Central Bank can commit to keep policy rates low for an extended period and enter into refinancing operations with extended maturity, thereby imposing a ceiling on interest rates at the corresponding horizon. Here, there is room for a monitoring of long-term rates: if $r_{\overline{0}}$ decreases (say, by the increase of $M_{\overline{0}}$), then the no-arbitrage relation between first-period long-term rates and second-period short-term rates implies that r_s must decrease for each s.[58]

this: as already said, interest-targeting policies are only partially equivalent to quantity-targeting ones, and require a specific inquiry ; second-period interest rates should depend upon first-period interest rates as well as commodity prices. A CME corresponding to such a monetary policy would therefore involve a fixed point involving both the monetary tools and the market clearing equations. We leave this for further research.

[57]Only above a certain threshold of injected money do we recover the dichotomy between the real and the nominal spheres, due to the boundedness of physical trades.

[58]Going further into the exploration of such an alternative policy would go beyond the scope of this paper, and is left for further research.

C) One difficulty in considering default together with money in our general set-up is that many standard properties of default-free general equilibrium theory with money fail. Prominent among them is the Quantity Theory of Money in period 1 (see Proposition 4.1 supra). This forces to find new arguments in several respects. Second, the classical non-arbitrage argument on the yield curve no more holds, so that it is not true, in general, that:

$$(1 + r_{\bar{0}}) \geq (1 + r_0) \min_s (1 + r_s). \tag{45}$$

This makes the existence proof more complicated, as the latter heavily relies on such a non-arbitrage relationship, e.g., in Dubey and Geanakoplos (2003b). Similarly, the fact that households may default on the loans of the Central Bank induces the failure of the following, otherwise standard, equality:

$$r_0 M_0 + r_{\bar{0}} M_{\bar{0}} + r_s M_s = \sum_h (m_0^h + m_s^h) \tag{46}$$

for every path $(0, s)$. Indeed, all we know is that, at equilibrium, no agent will end up with useless money, so that the whole quantity $\sum_h m_0^h + m_s^h$ is used to repay back interest rates. In the absence of default, (46) follows. But, if default is permitted as in this paper, all what follows is

$$r_0 M_0 + r_{\bar{0}} M_{\bar{0}} + r_s M_s \geq \sum_h m_0^h + m_s^h. \tag{47}$$

D) In this paper, we use two different institutional arrangements for the collateralization process: one, where the collateral vector for long-term loans, $\kappa_{\bar{0}}^h$, is borrower-specific and endogenous; the second, where the collateral vector for securities, κ_j, is exogenous. The reason why we use the endogenous arrangement for long-run loans relates to the interaction of money with default. Suppose, indeed, we were to allow for long-term loans collateralized by a fixed vector, $\kappa_{\bar{0}}$ (for each unit of credit extension). The interest rate would be given by:

$$1 + r_{\bar{0}} = \frac{\sum_h \mu_{\bar{0}}^h}{M_{\bar{0}} + \sum_h \tilde{\mu}_{\bar{0}}^h}$$

as follows from the (binding) market-clearing condition (27). The numerator, $\sum_h \mu_{\bar{0}}^h$, being bounded by the quantity of available collateral, $M_{\bar{0}} \to +\infty$ would imply $r_{\bar{0}} < 0$ at equilibrium. The endogenous formulation of collateral constraints avoids this pathology.

On the other hand, we impose an exogenous collateral level for financial assets for the sake of simplicity. Allowing for endogenous collateral levels on the financial market would make the existence proof more complicated, for we must have to find a bound for asset short-selling.

E) In asserting condition HDD, we have supposed that the asset functions are bounded in terms of asset prices.

Actually, this condition can be removed if there is no circularity in the definition of financial assets. To deal with the more general case, suppose that there exists a tower of assets, such that each asset is polynomially bounded in terms of prices, interest rates and prices of preceding assets. Thus, up to a reordering, we suppose that $A_{sj}^k = O(||p, r, \pi_{k'<k}||^{b_k})$, for $1 < k \leq K$. This assumption involves no loss of generality if there is a tower of asset and asset functions are semi-algebraic. Given the fact that finance industry invents new financial assets one after the other (first derivative on commodity prices, then derivatives of derivatives, and so on), the existence of a tower of assets is always assured in practice. If we order the existing assets by their date of invention, then the asset whose return is a function of other assets depends only on the prices of assets previously invented. So our definition essentially excludes unbounded yield of assets defined circularly (e.g., yield of asset 1 depends on the price of asset 2 and yield of asset 2 depends on the price of asset 1).

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A Appendix

This Appendix provides the proof of the existence theorem and of the properties of the yield curve.

A.1 The existence theorem

We prepare the proof with two lemmas.

Note, for $i \in \mathbf{S}^* \times \mathbf{L}$, $\mathbf{1}_i \in \mathbb{R}_+^{\mathbf{S}^* \times \mathbf{L}}$ the vector with 1 in component i and zero elsewhere, $\mathbf{1}$ is the vector $\mathbb{R}_+^{\mathbf{S}^* \times \mathbf{L}}$ with 1 in every component.

Let $K_* = \max_{\ell} \sum_{s,h} \mathbf{e}^h_{s\ell} + 1$. This is the maximum amount of each commodity that exist in our economy. Define $u^h_* := u^h(K_* \mathbf{1})$. This is the maximum utility that h can get, since the quantity available in the economy of each commodity in each state is less than K_*.

Let $\Box$ be the cube with sides of length K_* in $\mathbb{R}_+^{\mathbf{S}^* \times \mathbf{L}}$, that is the set $x \leq K_* \mathbf{1}$.

Lemma A.1 *There is H_* such that $u^h(H_* \mathbf{1}_i) > u^h_*$ for every component i of $\mathbf{S}^* \times \mathbf{L}$ and any h.*

Proof. Let $H^h > 0$ be chosen large enough so that, for H^h in any component:

$$u^h(0, ..., 0, H^h, 0, ..., 0) > u^h_*$$

The following argument (adapted from footnote 19 in Dubey and Geanakoplos (2003a)) proves that such an H^h exists, up to a redefinition of the utility function, outside the domain of the economy. Define $\tilde{u}^h : \mathbb{R}_+^{\mathbf{S}^* \times \mathbf{L}} \to \mathbb{R}$ by $\tilde{u}^h(y) := \inf\{L_x(y), x \in \Box\}$, where L_x is an affine function representing the supporting hyperplane to the graph of u^h at the point $(x, u^h(x))$. $\tilde{u}^h$ coincides with u^h on $\Box$ (by concavity of u^h), and there exists some H^h such that $\tilde{u}^h(0, ..., 0, H^h, 0, ..., 0) > u^h_*$ for H^h in any component. Then $H_* := \max_h H^h$ verifies the lemma. $\Box$

Lemma A.2 *There exists $\xi^h > 0$, such that if $x \in \mathbb{R}_+^{\mathbf{S}^* \times \mathbf{L}}$, $x \in \Box$ and $0 \leq \Delta \leq 1$ then $u^h(x + \Delta \mathbf{1}_i) - u^h(x) \geq \xi^h \Delta$ for every component i.*

Proof. Define $\xi^h = \min_{i,x}\{u^h(x + \mathbf{1}_i) - u^h(x) | x \in \mathbb{R}_+^{\mathbf{S}^* \times \mathbf{L}}, x \in \Box, i \in \mathbf{S}^* \times \mathbf{L}\}$. Because u^h is strictly increasing, $u^h(x + \mathbf{1}_i) - u^h(x)$ is positive. As the minimum is taken over a compact set, ξ^h is positive. Then $u^h(x + \Delta \mathbf{1}_i) \geq (1 - \Delta) u^h(x) + \Delta u^h(x + \mathbf{1}_i)$ by concavity of u^h. By construction of ξ^h, we therefore have the claimed lemma. $\Box$

Proof of Existence Theorem 1.

The main difficulty lies in the fact that *financial asset* prices may be zero.[59] As in Geanakoplos and Zame (2002), we introduce for each $\underline{\pi} > 0$, an auxiliary economy, $\mathcal{E}^{\underline{\pi}}$, which differs from $\mathcal{E}$ only in that asset promises are given by:

$$\mathbf{A}^{k;\underline{\pi}}_s(\eta_0, \eta_1) := \mathbf{A}^k_s(\eta_0, \eta_1) + \underline{\pi}.$$

[59] Given our rational expectations set-up, the price of any asset yielding a 0 return will necessarily be 0 at equilibrium. For instance, a call to purchase an ounce of gold at €800 will be priced 0 if, at equilibrium, the price of gold is always strictly less than €800.

We add a dummy player to this auxiliary economy. Within the augmented auxiliary economy, $\mathcal{E}^{\underline{\pi},\varepsilon}$, we prove existence of an active monetary equilibrium (part 1). We then remove the dummy player, by taking the limit as $\varepsilon \to 0^+$ and prove the existence of a monetary equilibrium in $\mathcal{E}^{\underline{\pi}}$ with asset prices that are bounded away from 0 (part 2). We finally take the limit $\underline{\pi} \to 0^+$ (part 3).

Part 1. Existence in $\mathcal{E}^{\underline{\pi},\varepsilon}$ with a dummy player.

For any $\varepsilon > 0$, we define a truncated generalized game $\Gamma^{\underline{\pi},\varepsilon}$ on a continuum player-set with types H. Each time h corresponds to, say, the unit interval $[0,1]$ of identical players, equipped with the restriction of the Lebesgue measure. Following Dubey and Geanakoplos (2003a, 2006a,b), we add a dummy player who puts up for sale ε units for sale of each instrument (commodities, assets, loans) except for assets, which is discussed below. Furthermore, she puts up ε units of money for purchase on every market. This external player fully delivers on her promises.

Recall that the functions A^k_j are polynomially dominated $A^k_j = O(||p,r||^b)$. The dummy player then put up for sale ε^{b+1} of each asset[60].

The other players act strategically, and prices form so as to clear every market (taking the dummy player into account). A type-symmetric Nash equilibrium (NE) of $\Gamma^{\underline{\pi},\varepsilon}$ will be called an ε-Monetary Equilibrium (ε-CME).[61] The payoff to any player h is her final utility $u^h(x^h)$.

We first construct truncated strategy sets in the auxiliary game $\Gamma^{\underline{\pi},\varepsilon}$. For any $\varepsilon > 0$, define $\Sigma^h_\varepsilon := \{\sigma^h \ : \ 0 \le \sigma^h \le 1/\varepsilon\}$, the ambient strategy space of type h where asset purchases and sales are bounded by $1/\varepsilon$. This completes the construction of the generalized market game $\Gamma^{\underline{\pi},\varepsilon}$. Since players can bid and supply on each side of each market, it can be interpreted as a double auction where only market orders are allowed (and not limit-price orders).

We define a point-to-set map (a correspondance) ψ from $\prod_{h\in\mathbf{H}} \Sigma^h_\varepsilon$ onto itself. Given an action profile $\sigma = (\sigma^h) \in \prod_h \Sigma^h_\varepsilon$, define the macrovariables $\eta_\varepsilon(\sigma) := (r, \pi, p)$, recursively as follows:

$$\frac{1}{1+r_n} := \frac{\varepsilon + M_n + \sum_h \tilde{\mu}^h_n}{\varepsilon + \sum_h \mu^h_n} \qquad (n^{\text{th}} \text{ loan market})$$

$$\pi^k := \frac{\varepsilon^2 + \sum_h \tilde{\alpha}^h_k}{\varepsilon^2 + \sum_h \alpha^h_k} \qquad (k^{\text{th}} \text{ asset})$$

$$p^\ell_s := \frac{\varepsilon + \sum_h \tilde{q}^h_{s\ell}}{\varepsilon + \sum_h q^h_{s\ell}} \qquad (\text{commodity } s\ell)$$

Finally, the subset of Σ^h_ε that is feasible for player h, given σ, is $\Sigma^h_\varepsilon \cap \Sigma^h_{\eta_\varepsilon(\sigma)}$.

Let $\psi^h(\sigma) = \arg\max u^h(\overline{\sigma}^h)$, where the maximum is taken over $\overline{\sigma}^h$ in the feasible subset of Σ^h_ε. Then $\psi = \prod_h \psi^h$.

[60] In case of a tower of assets, the quantity bought and sold depends on the asset. Recall that we imposed that $A^k_j = O(||p, r, \pi_{k'<k}||^{b_k})$. The dummy player then puts for sale ε^{β_k} of asset k, where the β_k are defined recursively by $\beta_1 = 1 + b_1, \beta_k = 1 + b_k \max_{k'<k} \beta_{k'}$.

[61] Throughout the proof, we confine ourselves to type-symmetric action profiles. By a slight abuse of notations, the action, σ^h, of type h will denote either the aggregate action, $\int_{[0,1]} \sigma^\tau d\lambda(d)$, or the action of a single, negligible, individual $\tau \in [0,1]$. The interpretation should be clear from the context.

Because, thanks to the introduction of the dummy player, all the standard convexity and continuity assumptions are satisfied, best-reply correspondences ψ_h have a closed graph and convex values. Thus, the standard Kakutani-fixed-point argument ensures that there exists a type-symmetric pure NE in the truncated generalized game $\Gamma^{\underline{\pi},\varepsilon}$. Choose a fixed point $\sigma(\varepsilon)$ and denote the macrovariables $\eta_\varepsilon(\sigma(\varepsilon))$ by $\eta^\varepsilon = (r^\varepsilon, \pi^\varepsilon, p^\varepsilon)$. This is an ε-CME.

Part 2. Dropping the dummy player.

In this part, we show that a limit of ε-CME, as $\varepsilon \to 0^+$ is a *bona fide* CME of $\mathcal{E}^{\underline{\pi}}$. From now on, we set $\varepsilon, \underline{\pi} < 1$. Let $\varepsilon \to 0$, up to a subsequence we can suppose that each component (and each ratio of these components) of $\sigma(\varepsilon)$ and η_ε converges (possibly to zero or infinity).

Step **1** $\sigma^h(\varepsilon)$ *maximizes* u^h *in* $\Sigma^h_{\eta^\varepsilon}$ *(and not just* $\Sigma^h_\varepsilon \cap \Sigma^h_{\eta^\varepsilon}$*, which is the case by construction).*

To prove the claimed result, we prove that all σ^h in $\Sigma^h_{\eta^\varepsilon}$ are bounded, independently of ε and $\underline{\pi}$.

By assumption, collateral requirements for each asset are non zero. Choose a constant $\mathbf{M}$ so large that, for each k, $\exists \ell \mathbf{M}\kappa^k_\ell \geq \overline{\mathbf{e}}_\ell := \sum_h \mathbf{e}^h_\ell$. Thus, to sell more of $\mathbf{M}$ units of asset $\mathbf{A}^k_s$ would require more collateral than is available in the entire economy. So $\alpha^h_k \leq \mathbf{M}$. By physical constraints (22), we have $q^h_{0\ell} \leq e^h_{0\ell}$ $\forall \ell \in \mathbf{L}$, so that q^h_0 is also bounded. By (13), $\tilde{\mu}^h_0$ and $\tilde{\mu}^h_{\overline{0}}$ are also bounded.

By construction of the interest rate r^ε_0, we have $\frac{\mu^h_0}{1+r^\varepsilon_0} \leq \varepsilon + M_0 + \sum_h \tilde{\mu}^h_0$ and the same for $\overline{0}$. Now by (14), we have $\sum_\ell \tilde{q}^h_{0\ell} + \sum_k \tilde{\alpha}^h_k \leq m^h_0 + \varepsilon + M_0 + \sum_h \tilde{\mu}^h_0 + \varepsilon + M_{\overline{0}} + \sum_h \tilde{\mu}^h_{\overline{0}} \leq 2\overline{M}$. So that $\tilde{q}^h_0$ and $\tilde{\alpha}^h_k$ are bounded.

By construction of the prices p^ε_0, $p^{\varepsilon\ell}_0 q^h_{0\ell} \leq \varepsilon + \sum_{g\in H} \tilde{q}^g_{0\ell}$, which is bounded since each $\tilde{q}^g_{0\ell}$ is. The same apply for $\pi^{\varepsilon k}\alpha_k$. Thus the rhs of (16) is bounded. Recalling that either μ^h_0 or $\tilde{\mu}^h_0$ is null, it follows that μ^h_0 is bounded, and thus r_0 as well (see next step for a detailed proof). This ensures that in every case, $\Delta(16)$ is bounded.

The same proof applies for (17) and shows $r_{\overline{0}}\frac{\mu^h_{\overline{0}}}{1+r_{\overline{0}}}$ is bounded and so $\mu^h_{\overline{0}}$ is because $\frac{\mu^h_{\overline{0}}}{1+r_{\overline{0}}}$ already was. Thus $r_{\overline{0}}$ is (see next step for a detailed proof). This ensures that in every case, $\Delta(17)$ is bounded.

By construction of the prices p^ε_0, $\tilde{q}^h_{0\ell}/p^{\varepsilon\ell}_0 \leq \varepsilon + \sum_{g\in H} q^g_{0\ell}$, which is bounded since each $q^g_{0\ell}$ is. Then $\kappa^h_{\overline{0}\ell}$ is bounded by (15) and so is $x^h_{\overline{0}}$ by (23).

In state s, the proof follows the same lines. We first prove that q^h_s is bounded by (24) (recall that g^h_s is linear), and that $\tilde{\mu}^h_s$ is bounded by (18), then, by construction of r^ε_s, $\frac{\mu^h_s}{1+r^\varepsilon_s}$ is bounded. By (19) we can then prove that $\tilde{q}^h_s$ and, by construction of prices p^ε_s, that $p^{\varepsilon\ell}_s q_{s\ell}$ are bounded, and, by (20), that $\mathbf{A}^{k,\underline{\pi}}_s \alpha^h_k$ are bounded.

To prove that μ^h_s are bounded, it suffices to show that the rhs of (21) is bounded i.e. that $\mathbf{A}^{k,\underline{\pi}}_s \frac{\tilde{\alpha}^h_k}{\pi^{\varepsilon k}}$ is bounded. But, by construction of $\pi^{\varepsilon k}$, we have:

$$\mathbf{A}^{k,\underline{\pi}}_s \frac{\tilde{\alpha}^h_k}{\pi^{\varepsilon k}} = \varepsilon^{b+1}\mathbf{A}^{k,\underline{\pi}}_s \frac{\tilde{\alpha}^h_k}{\varepsilon^{b+1} + \sum_h \tilde{\alpha}^h_k} + (\sum_h \mathbf{A}^{k,\underline{\pi}}_s \alpha^h_k)\frac{\tilde{\alpha}^h_k}{\varepsilon^{b+1} + \sum_h \tilde{\alpha}^h_k}$$

Of course $\frac{\tilde{\alpha}^h_k}{\varepsilon^2 + \sum_h \tilde{\alpha}^h_k} \leq 1$, so the second term is bounded, because we have already proven that $\mathbf{A}^{k,\underline{\pi}}_s \alpha^h_k$ are bounded. It remains to show that $\varepsilon^{b+1}\mathbf{A}^{k,\underline{\pi}}_s$ is bounded. We have precisely adapted the behavior of the dummy player on assets to be the case. Indeed, $\mathbf{A}_s^{k,\underline{\pi}} = \underline{\pi} + \sum_j p_{sj} A_j^k(\eta^\varepsilon)$ and we have supposed that $A_j^k(\eta^\varepsilon) = O(||p^\varepsilon, r^\varepsilon||^b)$. Because of the construction of prices and interest-rates, we have $p^\varepsilon, r^\varepsilon = O(1/\varepsilon)$. Thus $\mathbf{A}_s^{k,\underline{\pi}} = O(1/\varepsilon^{b+1})$. So the conclusion follows[62].

We proved that σ^h is bounded, independently of ε and $\underline{\pi}$ (we put bounds on q^h by commodity scarcity, on α_k^h by collateral requirements, and then on the other variables by bounding, at each step, the quantity of money available). As the extra bounds of $1/\varepsilon$ are irrelevant, for sufficiently small ε, it follows that $\sigma^h(\varepsilon)$ maximizes u^h in $\Sigma^h_{\eta^\varepsilon}$.

Step 2 *Interest rates, r_n^ε, are non-negative and bounded, for every n.*

Since the μ_n^h are bounded and $1 + r_n^\varepsilon = \frac{\varepsilon + \sum_h \mu_n^h}{\varepsilon + M_n + \sum_h \tilde{\mu}_n^h}$, r_n^ε is bounded. Furthermore, $r_n^\varepsilon \geq 0$, since, if the contrary, player h could improve her profile σ^h by borrowing more money, spending a little on commodity, inventoried the money to pay back the extra-loan.

Step 3 *The ratio of commodity prices don't go to infinity: the ratios $p_s^{\varepsilon\ell}/p_s^{\varepsilon\ell'}$ are bounded for every $s \in \mathbf{S}^*$ and every $\ell, \ell' \in \mathbf{L}$, as are the ratio $p_0^{\varepsilon\ell}/p_s^{\varepsilon\ell'}$ are bounded for every $s \in \mathbf{S}$ and every $\ell, \ell' \in \mathbf{L}$.*

Suppose some $p_s^{\varepsilon\ell}/p_s^{\varepsilon\ell'} \to \infty$. Take h with $e_{s\ell}^h > 0$. Let set her apart $\Delta e_{s\ell}^h$ of his endowment and scaling down her actions by $1 - \Delta$. Her utility decreases by at most $\Delta(u_*^h - u^h(0))$ and she has still at least $\Delta e_{s\ell}^h$. Let h borrow more money on M_s, increasing μ_s^h by $\Delta p_s^{\varepsilon\ell} e_{s\ell}^h$ (possible if ε is sufficiently small, because the extra boundaries $1/\varepsilon$ are not binding by step 1.), spending the money to purchase and consume ℓ' (in quantity $\Delta p_s^{\varepsilon\ell} e_{s\ell}^h/(p_s^{\varepsilon\ell'}(1 + r_s^\varepsilon))$, selling $\Delta e_{s\ell}^h$ to pay back the loan. Choosing Δ small enough (depending on ε) so that $\Delta p_s^{\varepsilon\ell} e_{s\ell}^h/(p_s^{\varepsilon\ell'}(1 + r_s^\varepsilon)) < 1$, we can apply lemma (A.2). The increase in h's utility is at least:

$$\Delta \left(\xi^h \frac{p_s^{\varepsilon\ell} e_{s\ell}^h}{p_s^{\varepsilon\ell'}(1 + r_s^\varepsilon)} - [u_*^h - u^h(0)] \right)$$

Since by assumption $p_s^{\varepsilon\ell}/p_s^{\varepsilon\ell'} \to \infty$, the utility increase becomes positive because r_s^ε is bounded. The proof is the same for $p_0^{\varepsilon\ell}/p_s^{\varepsilon\ell'}$, except that h first sell a little of good ℓ in period 0 and inventoried the money in period s to buy ℓ'.

Step 4 *$\pi^{\varepsilon k}/p_s^{\varepsilon\ell}$ remains bounded for all assets k, all commodities ℓ and every state s.*

Suppose some $\pi^{\varepsilon k}/p_s^{\varepsilon\ell'} \to \infty$. Given asset k, take some household h that has initial endowment of each commodity that is required as collateral in order to sale k –i.e. for each ℓ such that $\kappa_\ell^k > 0$, $e_\ell^h > 0$. (If no such player exists, then asset k will never be traded and can be disregarded.) Let h scale down her actions by $1 - \Delta$. Her utility decreases by at most $\Delta(u_*^h - u^h(0))$ and she has still at least $\Delta e_{s\ell}^h$ for each commodity that makes the collateral of asset k, she thus can sell at least $\alpha_k^h = \Delta \min_\ell \{e_{s\ell}^h/\kappa_\ell^k | \kappa_\ell^k > 0\}$ more of asset k and get $\pi^{\varepsilon k}\alpha_k$ of money. If $s = 0$, she can

[62]The proof is the same in the case of a tower of assets, given the behavior of the dummy player we have constructed.

increase her borrowing μ_0^h by $\pi^{\varepsilon k}\alpha_k$, spend the money to obtain more ℓ' in quantity $\pi^{\varepsilon k}\alpha_k/(p_0^{\varepsilon\ell'}(1+r_0^\varepsilon))$, sell the asset to pay back her loan. As she have the collateral, there is no problem for delivery on this asset. The increase in h's utility is at least

$$\Delta\left(\xi^h \frac{\pi^{\varepsilon k}\min\{e_{s\ell}^h/\kappa_\ell^k\}}{p_s^{\varepsilon\ell'}(1+r_0^\varepsilon)} - [u_*^h - u^h(0)]\right)$$

Since by assumption $\pi^{\varepsilon k}/p_s^{\varepsilon\ell'} \to \infty$, the utility increase becomes positive because r_0^ε is bounded. The proof is simpler if $s \in \mathbf{S}$, because h can sell the extra amount of asset k in period 0, inventoried the money in period 1, and then buy more of ℓ'.

Step **5** *Prices are bounded away from* 0*: there exists* $\underline{p} > 0$ *such that* $p_s^{\varepsilon\ell} > \underline{p}$ *for every commodity* ℓ *and every state* $s \in \mathbf{S}^*$.

Suppose first that we are in environment A: $\sum_h m_0^h > 0$. Take h with $m_0^h > 0$. Let H_* as in lemma (A.1). Now, we claim that $p_s^{\varepsilon\ell} \geq \frac{m_0^h}{H_*}$. Otherwise, agent h could spend her money in order to buy H^* units of commodity ℓ, thus obtaining a final utility $\tilde{u}^h(0,...,0,H_*,0,...0)$ higher than $u^h(K_*,...,K_*)$. A contradiction since K_* is the maximum utility the agent h can get.

Suppose that we are in environment B, $\sum_h m_0^h = 0$ but $M_{\overline{0}} > 0$. Suppose that a price of a commodity is not bounded away. Up to a subsequence, it tends to 0. Thanks to step 3, all prices tends to 0. Because of the collateral constraints (15) and the construction of interest rate, we have the following relation:

$$p_0 \cdot \sum_h \kappa_{\overline{0}}^h = \sum_h \frac{\mu^h}{1+r_{\overline{0}}} = \varepsilon + M_{\overline{0}}$$

Because $M_{\overline{0}} > 0$, at least one of the $\kappa_{\overline{0}\ell}^h$ must tend to ∞, because all prices tend to 0, as ε tends to 0. But this contradicts the hypothesis that there is a finite amount of goods in the economy.

So prices are bounded from below in period 0. By step 3, it is also the case in all state $s \in \mathbf{S}$.

Denote $r_n = \lim r_n^\varepsilon$ for $n \in N$ (the limit exists up to a subsequence by step 1.).

Step **6** *Let* $s \in \mathbf{S}$*. Then* $r_s \leq \hat{m}_s/M_s$.

We sum over h the liquidity constraints to get, introducing the construction of r_n^ε, $\pi^{\varepsilon k}$ and p_s^ε, we have the inequality:

$$r_0^\varepsilon\left(M_0 + \varepsilon\frac{r_0^\varepsilon}{1+r_0^\varepsilon}\right) + r_s^\varepsilon\left(M_s + \varepsilon\frac{r_s^\varepsilon}{1+r_s^\varepsilon}\right) + r_{\overline{0}}^\varepsilon\left(M_{\overline{0}} + \varepsilon\frac{r_{\overline{0}}^\varepsilon}{1+r_{\overline{0}}^\varepsilon}\right)$$

$$\leq \sum_h (m_0^h + m_t^h) + \varepsilon\sum_\ell(1-p_0^{\varepsilon\ell}) + \varepsilon\sum_\ell(1-p_s^{\varepsilon\ell}) + \sum_k \varepsilon^{b+1}(1-\pi^{\varepsilon k})$$

$$+ \sum_k \mathbf{A}_s^{k,\underline{\pi}}\varepsilon^{b+1}\frac{\pi^{\varepsilon k}-1}{\pi^{\varepsilon k}} + M_{\overline{0}} + \varepsilon\frac{r_{\overline{0}}^\varepsilon}{1+r_{\overline{0}}^\varepsilon} - K_s$$

Thus

$$r_0^\varepsilon M_0 + r_s^\varepsilon M_s + r_{\overline{0}}^\varepsilon M_{\overline{0}} \leq \sum_h (m_0^h + m_t^h + \varepsilon(L+K) + \sum_k \mathbf{A}_t^{k,\underline{\pi}} \varepsilon^{b+1} + [M_{\overline{0}} + \varepsilon - K_s]$$

So the interest payment are covered by outside money, money created by the dummy player on goods and assets markets, money created by the dummy players on delivery (note that $\mathbf{A}_t^{k,\underline{\pi}} \varepsilon^{b+1}$ is bounded, because $\mathbf{A}_s^{k,\underline{\pi}} = O(1/\varepsilon^{b+1})$ thanks to the HDD hypothesis), and money created by default.

Thus, taking limits, we have:

$$r_0 M_0 + r_s M_s + r_{\overline{0}} M_{\overline{0}} \leq \sum_h (m_0^h + m_t^h) + [M_{\overline{0}} - K_s]$$

Thus $r_s M_s \leq \hat{m}_s$.

Step 7 *Let $s \in \mathbf{S}$, then $p_s^{\varepsilon\ell}$ is bounded for a commodity ℓ (and hence for all by step 3).*

Suppose that, up to a subsequence, $p_s^{\varepsilon\ell} \to \infty$ (for one, and hence, all ℓ). By construction of $p_s^{\varepsilon\ell}$, $p_s^{\varepsilon\ell} = \frac{\varepsilon + \sum_h \tilde{q}_{s\ell}^h}{\varepsilon + \sum_h q_{s\ell}^h}$. Because $\tilde{q}_{s\ell}^h$ are bounded, every $q_{s\ell}^h$ must go to 0. Therefore the final limit allocation of good x is in X_s (no trade is involved in state s). Let $||.||$ be a norm on the space of commodities prices and define $\hat{p}_\ell = \lim_{\varepsilon \to 0} p_\ell^\varepsilon / ||p^\varepsilon||$.

For each agent $h \in \mathbf{H}$, define a utility of trade τ in state s by $v^h(\tau) = u^h(x^h + \tau^*(\tau, r_s))$ where $\tau^*(\tau, r_s) \in \mathbb{R}^{\mathbf{S}^* \times \mathbf{L}}$ is given by $\tau_{t\ell}^* = 0$ if $t \in \mathbf{S}^* \backslash \{s\}$, $\tau_{s\ell}^* = \tau_\ell$ if $\tau_\ell < 0$, $\tau_{s\ell}^* = \tau_\ell / (1 + r_s)$ if $\tau_\ell \geq 0$. This defines a pure exchange L-goods economy in state s with utilities v^h, endowments x_s^h and prices $\hat{p}$. We want to prove that no-trade constitutes a Walrasian equilibrium of this economy.

Suppose not. Then for a player h there exists τ^h, such that $\hat{p} \cdot \tau^h = 0$ and $u^h(x^h + \tau^*(\tau, r_s)) > u^h(x^h)$. Then h would buy $\tilde{q}_{s\ell}^h + p_{s\ell}[\tau_\ell^h]^+ / (1 + r_s)$ instead of $\tilde{q}_\ell^h$ by taking extra short loan and selling $q_{s\ell}^h + [\tau_\ell^h]^-$ instead of $q_{s\ell}^h$. Up to an infinitesimal adjustment when $\varepsilon \to 0$, h can reimburse her extra loan with her commodities sales, and end up with a greater utility —which contradicts the assumption that we were at a ε-CME. Hence no-trade constitutes a Walrasian equilibrium of the pure exchange economy.

By definition, this proves that the spot allocation x is r_s-Pareto-optimal in state s. Thus, gains-to-trade are upper-bounded: $\gamma_s(x) \leq r_s$. But, by the step 6 $r_s \leq \hat{m}_s / M_s$. Which contradicts the gains-to-trade hypothesis $\hat{m}_s / M_s < \gamma_s(x)$. Hence prices remain bounded.

Step 8 *All prices are bounded.*

With step 7, prices of commodity are bounded in every state $s \in \mathbf{S}$. But by step 3, this is also the case for 0. By step 4, the price are asset are bounded. So we have a regular ME-equilibrium of $\mathcal{E}^{\underline{\pi}}$

Part 3. We then take the limit $\underline{\pi} \to 0^+$ to obtain a CME of our economy. It is possible because we have bounded selling and buying independently of $\underline{\pi}$, thanks to the liquidity constraints. □

A.2 Proofs of properties of CME

Proof of proposition 4.1.

(i) Summing (14) over h yields:

$$\sum_h \left[\tilde{\mu}_0^h + \tilde{\mu}_{\bar{0}}^h + \sum_k \tilde{\alpha}_k^h + \sum_\ell \tilde{q}_{0\ell}^h + \Delta(14)^h\right] = \sum_h m_0^h + \frac{1}{1+r_0}\sum_h \mu_0^h + \frac{1}{1+r_{\bar{0}}}\sum_h \mu_{\bar{0}}^h.$$

The conclusion follows by (27), (28) and (29).

When $r_0 > 0$, we prove that the liquidity constraint (14) is binding, so $\Delta(14)^h = 0$ for all agents h. Suppose $\Delta(14) > 0$. One of the three terms $\Delta(13)$, $\frac{\mu_0^h}{1+r_0}$ or $\frac{\mu_{\bar{0}}^h}{1+r_{\bar{0}}}$ must be positive. If $\Delta(13) > 0$, individual h can increase her deposits on 0 by ϵ; if $\frac{\mu_0^h}{1+r_0} > 0$, individual h can reduce μ_0^h by an amount of $\epsilon(1+r_0)$. In both cases, this increases $\Delta(16)$ by ϵr_0, positive by assumption, and so the right-hand side of (19). Individual h can thus increase her final allocation in consumption commodities in state s. A contradiction. So if $\Delta(14) > 0$ one must have $\Delta(13) = 0$ and $\mu_0^h = 0$, so that the individual h has only borrowed long-term money to spend on commodities and asset. But the collateral constraint (15) forces her to spend all the money on commodities. Hence $\Delta(14) = 0$, which contradicts the hypothesis. So $\Delta(14) = 0$ in every case.

(ii) The second inequality obtains similarly by summing (19) over h. The equality follows, since, as we now show, (19) must be binding when $r_s > 0$.

Suppose, first, that $\Delta(20) = 0$. We claim that it is always possible to choose the players' action so that $\Delta(19) = 0$. Indeed take, on the contrary, a player h with $\Delta(19) > 0$. Then, by assumption $p_s \cdot q_s^h - \sum_k \mathbf{A}_s^k \alpha_k^h = -\Delta(19) < 0$. We split $\mathbf{L}$ in $\mathbf{L}_+ = \{\ell | q_{\ell s}^h \geq \sum_k \kappa_\ell^k \alpha_k^h\}$ and $\mathbf{L}_- = \{\ell | q_{\ell s}^h < \sum_k \kappa_\ell^k \alpha_k^h\}$. $\mathbf{L}_-$ in non void, otherwise $p_s \cdot q_s^h \geq p_s \cdot \sum_k \kappa^k \alpha_k^h \geq \sum_k \mathbf{A}_s^k \alpha_k^h$, impossible by assumption.

Define

$$\gamma = \frac{-\Delta(19)}{\sum_{\ell \in \mathbf{L}_-} p_s^\ell (q_{s\ell}^h - \sum_k \kappa_\ell^k \alpha_k^h)}$$

Obviously $\gamma > 0$. But $\sum_{l \in \mathbf{L}_+} p_s^\ell q_{s\ell}^h + \sum_{l \in \mathbf{L}_+} p_s^\ell \kappa_\ell^k \alpha_k^h \geq p_s \cdot \sum_k \kappa^k \alpha_k^h \geq \sum_k \mathbf{A}_s^k \alpha_k^h = \sum_{l \in \mathbf{L}_+} p_s^\ell q_{s\ell}^h + \sum_{l \in \mathbf{L}_-} p_s^\ell q_{s\ell}^h + \Delta(19)$, thus $\sum_{\ell \in \mathbf{L}_-} p_s^\ell (\sum_k \kappa_\ell^k \alpha_k^h - q_{s\ell}^h) \geq \Delta(19)$, so $\gamma \leq 1$.

We redefine for $\ell \in \mathbf{L}_-$ the selling actions $q_{s\ell}^h := (1-\gamma) q_{s\ell}^h + \gamma \sum_k \kappa_\ell^k \alpha_k^h$ and the buying actions $\tilde{q}_\ell^h := \tilde{q}_\ell^h + p_s^\ell \gamma (\sum_k \kappa_\ell^k \alpha_k^h - q_{s\ell}^h)$.

Being a convex combination of the previous (feasible) action and the feasible action that sells entirely the stored collateral in ℓ, the selling action satisfies the physical constraints.

With these actions, we now have $\Delta(20) = 0$ and $\Delta(19) = 0$. The quantity available for agent h is unchanged, so that final utility levels are unchanged. As a consequence, from now on, we shall always assume that $\Delta(20) = 0 \Rightarrow \Delta(19) = 0$.

Suppose then that $\Delta(19) > 0$. Set $\epsilon = \min(\Delta(20), \Delta(19))$, by assumption $\epsilon > 0$. At least one of $\Delta(18)$ or $\frac{\mu_s^h}{1+r_s}$ is positive. So, let individual h increase her deposits $\tilde{\mu}_s^h$ by $\frac{\Delta(19)}{1+r_s}$ or reduce her short loan μ_s^h on s by ϵ. She spends $r_s \epsilon/(1+r_s)$ (which is positive since $r_s > 0$) more on final commodities. Thus $\Delta(19)$ decreases by ϵ and thus, by construction of ϵ, (19) and (20) are still satisfied. Right-hand side of (21) is decreased by ϵ, but so his left-hand side, because h has reduced her short loan or increased her deposits by this amount. So h has increased her utility in state s. A contradiction.

This proves that (19) is binding. One then sums (19) up to (13) over h and uses (27), (29) to obtain the desired conclusion.

For the third equation one just notices that $\Delta(21) = 0$. Indeed, if h has some money left at the end of s, she could have borrowed more, consumed more, and still been able to re-pay her loan. One then sums (21) over h and uses (27), (28) and $\Delta(19) = 0$ to obtain the desired conclusion. Alternatively, one can use (39) (which also relies on $\Delta(21) = 0$) to derive the third equation from the second. □